Math 574 Approximation Theory Qualifying Exam

Spring 2011

1. Let $X : C([0,1]) \to P_1$, where P_1 is the space of polynomials of degree at most one, be defined as

$$(Xf)(x) = 2\int_0^{1/2} f(t)dt + \left(x - \frac{1}{4}\right)[f(1) - f(0)]\,,\ 0 \leq x \leq 1.$$

   (a) Show that X is a projection.

   (b) Prove that $\|X\| \leq \frac{5}{2}$, where $\|X\| = \sup\{\|Xf\|_\infty : \|f\|_\infty \leq 1\}$.

   (c) Use Lebesgue's inequality to prove that

$$\|f - Xf\|_\infty \leq \frac{7}{2}\|f - p\|_\infty$$

   holds for all polynomials $p \in P_1$.

2. Let $f \in C^{k+1}([a,b])$ and assume that $f^{(k)}$ increases strictly monotonically on $[a,b]$. Further, let $a \leq x_0 < x_1 < \ldots < x_m \leq b$, where $m > k$. Prove that the sequence of divided differences $\left\{f\left[x_j, x_{j+1}, \ldots, x_{j+k}\right]\right\}_{j=0}^{m-k}$ increases strictly monotonically.

3. Let p_n be a polynomial of degree at most n that interpolates a function f at $x_0 \leq x_1 \leq \ldots \leq x_n$.

   (a) Prove that

$$f(x) = p_n(x) + (x - x_0)(x - x_1)\cdots(x - x_n)f[x_0, x_1, \ldots, x_n, x].$$

   (b) Let $[a,b] = [x_0, x_n]$ and assume that $f \in C^{n+1}([a,b])$. Use the result of Part (a) to prove Taylor's Theorem:

$$f(x) = \sum_{i=0}^{n} \frac{(x - x_0)^i}{i!} f^{(i)}(x_0) + \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

   for some $\xi \in (a,b)$.

4. Find the best minimax approximation $p_{n-1}^* \in P_{n-1}$ of $p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ on the interval $[-1,1]$.

5. Let B be a normed linear space. Let A be a finite-dimensional subspace of B and let $X : B \to A$ be a linear projection whose norm is defined as

$$\|X\| = \sup\{\|Xf\| : \|f\| \le 1\}.$$

   (a) Prove that $\|X\| \ge 1$.

   (b) Assume that B is the inner product space equipped with the norm $\|f\| = \sqrt{(f,f)}$ and let $p^* \in A$ be the least squares approximation of $f \in B$. It is known that $p^* = Xf$, where X is a linear projection from B to A. Show that $\|X\| = 1$.
   Hint: $\|f\|^2 = \|f - p^*\|^2 + \|p^*\|^2$

6. Let $f \in C^{2k+2}([a,b])$ and let $w(x)$ be the weight function on $[a,b]$. Consider the Gauss quadrature formula

$$\int_a^b w(x)f(x)dx = \sum_{i=0}^{k} c_i f(x_i) + Rf. \tag{1}$$

   (a) Prove that

$$Rf = \frac{f^{(2k+2)}(\xi)}{(2k+2)!}\int_a^b w(x)\prod_{j=0}^{k}(x - x_j)^2 dx.$$

   (b) Show that $c_i = \int_a^b w(x)\left[L_i(x)\right]^2 dx$, where $L_i(x)$ is the ith Lagrange fundamental polynomial based on the nodes $x_0, x_1, \ldots, x_k$.

   (c) Determine the nodes x_i, $0 \le i \le k$, of the Gauss quadrature formula (1) if $[a,b] = [-1,1]$ and $w(x) = \frac{1}{\sqrt{1-x^2}}$.

7. Prove that the Chebyshev polynomials of the first kind $T_n(\cos\theta) = \cos(n\theta)$ are orthogonal on $[-1,1]$ with the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

   Hint: You may use the trigonometric identity $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$.