Math 575 Matrix Computations Qualifying Exam

Spring 2011

1. Let A be a 4×4 matrix to which we apply the following operations in the indicated order:
   (a) Subtract 3 times row two from row four.
   (b) Interchange columns one and three.
   (c) Delete row four.

   Write the result as a product of four matrices. Do <u>not</u> evaluate the resulting expression.

2. Let $v \in \mathbb{R}^m$ be a unit vector, i.e. $\|v\|_2 = 1$. The Householder transformation is defined as
$$Q = I - 2vv^T.$$
   (a) Show that $Q = Q^T = Q^{-1}$.
   (b) Find all the eigenvalues and associated eigenvectors of Q.

3. Let $D = \operatorname{diag}(d_1, d_2, \ldots, d_m)$. Show that:
   (a) $\|D\|_p = \max\limits_{1 \le i \le m} |d_i|$, $1 \le p \le \infty$.
   (b) $\sup\limits_{x \neq 0} \dfrac{x^T D x}{x^T x} = \max\limits_{1 \le i \le m} d_i$.

4. Consider the overdetermined linear system $Ax \approx b$, where $A \in \mathbb{R}^{m \times n}$ with $m > n$ and $b \in \mathbb{R}^m$. Suppose that $A = \widehat{Q}\widehat{R}$ is the reduced QR factorization of A. Show that the least squares solution x of $Ax \approx b$ satisfies
$$\widehat{R}x = \widehat{Q}^T b.$$

5. Let $A, Q \in \mathbb{R}^{m \times m}$, where Q is an orthogonal matrix. Show that
$$\kappa_2(QA) = \kappa_2(A),$$
   where $\kappa_2(\cdot)$ stands for the condition number of a matrix in the 2-norm.

6. Given
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$
how many singular values of A are equal to zero?

<u>Hint</u>: You do not need to compute the SVD to answer this question.

7. Let $A = U\Sigma V^T$ be the SVD of a matrix $A \in \mathbb{R}^{m\times n}$. Assume that $\operatorname{rank}(A) = r \le n$. Prove:
   (a) $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2$.
   (b) $\|A\|_2 = \sigma_1$.
   (c) $\kappa_2(A) = \sigma_1/\sigma_n$, provided $m = n = r$.
   (d) $|\det A| = \prod_{i=1}^{m} \sigma_i$, assuming that $A \in \mathbb{R}^{m\times m}$.
   (e) $A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$.
   (f) $A^{-1} = \sum_{i=1}^{m} \frac{v_i u_i^T}{\sigma_i}$, assuming that $A \in \mathbb{R}^{m\times m}$ and A is nonsingular.

8. Let $A \in \mathbb{C}^{m\times m}$ be hermitian positive definite and let $X \in \mathbb{C}^{m\times n}$, where $m \ge n$ and $\operatorname{rank}(X) = n$.
   (a) Prove that the matrix X^*AX is also hermitian positive definite.
   (b) Let A_n be the nth principal submatrix of A, i.e.
$$A_1 = [a_{1,1}],\ A_2 = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix},$$
   etc. Prove that for every $n = 1, 2, \ldots, m$, the matrix A_n is also hermitian positive definite.