MATH 519 QUALIFYING EXAM

FALL 2008

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let G be a group. Recall that an automorphism $\varphi : G \to G$ is said to be *inner* if there exists $g \in G$ such that $\varphi(x) = gxg^{-1}$ for all $x \in G$. Prove that the set of inner automorphisms forms a normal subgroup of the group of all automorphisms.

2. Let H and K be subgroups of a finite group G such that $|G : H|$ and $|G : K|$ are relatively prime. Show that $|G : H \cap K| = |G : H| \cdot |G : K|$ and $G = HK$.

3. Let G be a finite group. Show that if G has a normal subgroup N of order 3 that is not contained in the center of G, then G has a subgroup of index 2.

(*Hint. The group G acts on N by conjugation.*)

4. Let H be a Sylow p-subgroup of a finite group G. Let $K = N(H)$. (Normalizer). Prove or disprove: $K = N(K)$.

5. Let N_1, N_2 and N_3 be normal subgroups of a group G and assume that for $i \neq j$, $N_i \cap N_j = 1$, $N_iN_j = G$. Show that G is isomorphic to $N_1 \times N_1$ and G is abelian.

6. Let $\mathbb{F}_p$ be the finite field with p elements and let $GL_2(\mathbb{F}_p)$ denote the multiplicative group of all 2×2 invertible matrices with coefficients in $\mathbb{F}_p$. Then

$$|GL_2(\mathbb{F}_p)| = p^4 - p^3 - p^2 + p = p(p-1)^2(p+1).$$

(a) Prove that

$$P = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p\}$$

is a Sylow p-subgroup of $GL_2(\mathbb{F}_p)$.

(b) Let D denote the subgroup of all diagonal matrices in G. Show that D is contained in the normalizer $N_G(P)$.

(c) Prove that the number of Sylow p-subgroups of $GL_2(\mathbb{F}_p)$ is $p+1$.

(*Hint. Exhibit two distinct Sylow p-subgroups. In addition, use (b) and* $|D| = (p-1)^2$.)

7. Let R be a commutative ring with identity and let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be a chain of prime ideals of R. Show that $I = \cup_n I_n$ is a prime ideal of R.

8. Let R be the ring of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where a and b are real numbers. Prove that R is isomorphic to $\mathbb{C}$, the field of complex numbers.

9. Show that the noncommutative ring of $n \times n$ real matrices has no proper (two-sided) ideal.

10. Let F be a field and let $R = F[X, Y]$ be the ring of polynomials in X and Y with coefficients in F. Show that $M = (X + 1, Y - 2)$ is maximal and $P = (X + Y + 1)$ is prime. Is P maximal? Justify your answer.