MATH 519 QUALIFYING EXAM

SPRING 2008

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let G be a group with cyclic automorphism group $Aut(G)$. Prove that G is abelian.

2. Prove that every group of order 35 is cyclic.

3. Show that if the center of a group G is of index n in G, then every conjugacy class of G has at most n elements.

4. Classify all abelian groups of order 120 and list at least 3 non-abelian groups of order 120. Prove that the non-abelian groups you listed are nonisomorphic.

5. Let H be a proper subgroup of a finite group G. Prove that the union of the conjugates of H is not the whole group G.

6. Let N be a normal subgroup of the finite group G. A subgroup H is said to be a **complement** for N in G if $NH = G$ and $N \cap H = 1$.

(a) Show that all complements for N in G are isomorphic.

(b) If H is a complement for N in G, show that any conjugate of H is also a complement for N in G.

(c) If N has a complement in G that is a p-group for some prime p, prove that every Sylow p-subgroup of G contains a complement for N.

7. Let

$$S = \{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \}.$$

(a) Prove that S is a subring of the ring $M_2(\mathbb{Z})$ of 2×2-matrices with coefficients in $\mathbb{Z}$.

(b) Prove that the map $\varphi : \mathbb{Z}[\sqrt{2}] \to S$ defined by

$$\varphi(a + b\sqrt{2}) = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$$

is a ring homomorphism.

8. An element e of a ring S is called *idempotent* if $e^2 = e$. Note that in a product $R \times R'$ of rings with unity, the element $e = (1, 0)$ is idempotent. The object of this problem is to prove a converse for commutative rings.

(a) Prove that if e is an idempotent element of a ring with unity, then $e' = 1 - e$ is also idempotent.

(b) Let e be an idempotent element of a commutative ring with unity, S, and let $e' = 1 - e$. Prove that S is isomorphic to the product $(eS) \times (e'S)$.

9. Let R be a commutative ring, and let I be an ideal of the polynomial ring $R[x]$. Suppose that the lowest degree of a nonzero element of I is n and that I contains a monic polynomial of degree n. Prove that I is a principal ideal.

10. Let R be a commutative ring with 1 and M a maximal ideal of R. Suppose I, J are ideals of R such that $IJ \subseteq M$. Show that $I \subseteq M$ or $J \subseteq M$.