MATH 519 QUALIFYING EXAM

Fall, 2009

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let D_n be the dihedral group of order $2n$. Let H be a subgroup of D_n of odd order. Prove H is cyclic.

2. Let G be a group and suppose that the subgroups of G are totally ordered by inclusion (that is, for any two subgroups H, K of G, either $H \subset K$ or $K \subset H$).

(a) Show that every element of G has finite order.
(b) Show that there is a prime p such that the order of every element of G is a power of p.
(c) Prove, or give a counter-example to, the statement: G must be finite.

3. Let N be a cyclic normal subgroup of a group G. Show that $N \cap G' \subset Z(G')$, where G' is the commutator subgroup of G.

4. Let G, H be finite groups and let $\varphi : G \to H$ be an *onto* homomorphism. Show that for every Sylow p-subgroup P of H there exists a Sylow p-subgroup Q of G such that $\varphi(Q) = P$.

5. Let $n_p(G)$ be the number of Sylow p-subgroups of G (where p is a prime). Let K be a normal subgroup of G. Prove $n_P(G/K)$ divides $n_p(G)$.

Hint: Let P be a Sylow p-subgroup of G and work with $N = N_G(P)$ and $M = N_G(PK)$.

6. An abelian group A has a subgroup B such that $A/B \cong \mathbb{Z} \times \mathbb{Z}$. Prove that A is isomorphic to $B \times \mathbb{Z} \times \mathbb{Z}$.

7. Let R be a commutative ring and I, J and K ideals of R. Prove:

$$(I + J + K)(JK + KI + IJ) = (J + K)(K + I)(I + J).$$

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8. Let R be an integral domain and let F be its field of fractions. For $q \in F$ define

$$I_q = \{r \in R : rq \in R\}.$$

(a) Show that I_q is a non-zero ideal of R.
(b) Suppose $R = \mathbb{Z}[\sqrt{-3}]$ and $q = (1 - \sqrt{-3})/2$. Show that I_q is not a principal ideal.
Hint: Use the fact that the absolute value on $\mathbb{C}$ is multiplicative.

9. Let R be an integral domain. A non-zero, non-unit $s \in R$ is called *special* if for all $a \in R$ there exist $q, r \in R$ such that

$$a = sq + r \qquad r = 0 \quad \text{or} \quad r \text{ is a unit.}$$

If $s \in R$ is special then show (s) is a maximal ideal.

10. Let R be a PID and F its field of fractions. Suppose S is a ring with $R \subset S \subset F$.
(a) Show that all elements of S can be written as a/b, where $a, b \in R$ and $1/b \in S$.
(b) Show that S is a PID.