Ph.D. Qualifying Examination in Statistics
4:30–8:30 Tuesday, August 28, 2012

1. Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf

$$f(x) = \theta(1+x)^{-(1+\theta)}, \quad x > 0,$$

where $\theta > 0$.

a) Find a sufficient statistic for θ.

b) Find a Cramer-Rao lower bound for unbiased estimators of $g(\theta) = \log(\theta)$.

2. Let $Y_1, ..., Y_n$ be independent identically distributed (iid) random variables from a distribution with probability density function (pdf)

$$f(y) = \frac{2}{\sigma\sqrt{2\pi}} \frac{1}{y^2} \exp\left(\frac{-1}{2\sigma^2 y^2}\right)$$

where $y > 0$ and $\sigma > 0$.

a) Find the maximum likelihood estimator (MLE) of σ^2.

b) Find the MLE of σ.

3. Let $Y_1, ..., Y_n$ be iid from a one parameter exponential family with pdf or pmf $f(y|\theta)$ with complete sufficient statistic $T(\boldsymbol{Y}) = \sum_{i=1}^n t(Y_i)$ where $t(Y_i) \sim \theta X$ and X has a known distribution with known mean $E(X)$ and known variance $V(X)$. Let $W_n = cT(\boldsymbol{Y})$ be an estimator of θ where c is a constant.

a) Find the mean square error (MSE) of W_n as a function of c (and of n, $E(X)$ and $V(X)$).

b) Find the value of c that minimizes the MSE. Prove that your value is the minimizer.

c) Find the uniformly minimum variance unbiased estimator (UMVUE) of θ.

4. Let $X_1, ..., X_n$ be a random sample from a Poisson distribution with mean θ.

a) Show that $T = \sum_{i=1}^n X_i$ is complete sufficient statistic for θ.

b) For $a > 0$, find the uniformly minimum variance unbiased estimator (UMVUE) of $g(\theta) = e^{a\theta}$.

c) Prove the identity:

$$E\left[2^{X_1}|T\right] = \left(1 + \frac{1}{n}\right)^T.$$

5. Let $Y_1, ..., Y_n$ be independent identically distributed random variables with pdf

$$f(y) = e^y I(y \geq 0) \; \frac{1}{\lambda} \; \exp\left[\frac{-1}{\lambda}(e^y - 1)\right]$$

where $y > 0$ and $\lambda > 0$.

a) Show $W = e^Y - 1 \sim \frac{\lambda}{2}\chi^2_2$.

b) What is the UMP (uniformly most powerful) level α test for
$H_0 : \lambda = 2$ versus $H_1 : \lambda > 2$?

c) If $n = 20$ and $\alpha = 0.05$, then find the power $\beta(3.8386)$ of the above UMP test if $\lambda = 3.8386$. Let $P(\chi^2_d \leq \chi^2_{d,\delta}) = \delta$. The tabled values below give $\chi^2_{d,\delta}$.

d	δ							
	0.01	0.05	0.1	0.25	0.75	0.9	0.95	0.99
20	8.260	10.851	12.443	15.452	23.828	28.412	31.410	37.566
30	14.953	18.493	20.599	24.478	34.800	40.256	43.773	50.892
40	22.164	26.509	29.051	33.660	45.616	51.805	55.758	63.691

6. Consider independent random variables $X_1, ..., X_n$, where $X_i \sim \mathrm{N}(\theta_i, \sigma^2)$, $1 \leq i \leq n$, and σ^2 is known.

a) Find the most powerful test of

$$H_0 : \theta_i = 0, \forall i, \text{ versus } H_1 : \theta_i = \theta_{i0}, \forall i,$$

where θ_{i0} are known. Derive (and simplify) the exact critical region for a level α test.

b) Find the likelihood ratio test of

$$H_0 : \theta_i = 0, \forall i, \text{ versus } H_1 : \theta_i \neq 0, \text{ for some } i.$$

Derive (and simplify) the exact critical region for a level α test.

c) Find the power of the test in (a), when $\theta_{i0} = n^{-1/3}, \forall i$. What happens to this power expression as $n \to \infty$?