MATH 501 QUALIFYING EXAM | **Sept. 24 (Thur.), 1998**
TIME: 11:00am - 3:00pm | LOCATION: Neckers 356
YOUR NAME: | YOUR: ID #:

Show all of your work. Also turn in all of your scratch papers.

1 (10pts) Give an example of a measurable space $(X, \mathbb{X})$, and a function f on X to $\mathbb{R}$ which is not $\mathbb{X}$-measurable, but is such that the functions $|f|$ and f^2 are $\mathbb{X}$-measurable.

2 (10pts) If $f \in L_p$, $1 \leq p < \infty$, and if $E = \{x \in X : |f(x)| \neq 0\}$, then E is σ-finite.

3 (15pts) Let $(X, \mathbb{X}, \mu)$ be a finite measure space, f is $\mathbb{X}$-measurable. For each $n = 1, 2, \cdots$, define the subset $E_n = \{x \in X : (n-1) \leq |f(x)| < n\}$. Prove that $f \in L_p$ if and only if

$$\sum_{n=1}^{\infty} n^p \mu(E_n) < +\infty$$

where $1 \leq p < \infty$.

4 (15pts) If $\{f_n\}$ is a sequence in $L(X, \mathbb{X}, \mu)$ which converges uniformly on X to a function f, and if $\mu(X) < +\infty$, then

$$\int f d\mu = \lim \int f_n d\mu.$$

#5 (20pts) Let $(X, \mathbb{X}, \mu)$ be a finite measure space and $1 < p < \infty$. Let G be a bounded linear functional defined on $L_p(X, \mathbb{X}, \mu)$ such that

$$G(f) = \int_X g f d\mu \qquad \text{for all } f \in L_p$$

where the function $g \in M^+(X, \mathbb{X})$. Prove that $g \in L_q$ and $\|G\| = \|g\|_q$ here $1/p + 1/q = 1$.

6 (20pts) Let $1 < p < \infty$. If $f_n \to f$ in L_p-norm (where $f_n \in L_p$ and f is measurable). Prove that

a) $f \in L_p$;

b) if $g_n \to g$ in L_q where $1/p + 1/q = 1$ ($g_n \in L_q$ for $n = 1, 2, \cdots$), then $f_n g_n \to fg$ in L_1.

7 (10pts) Let λ and μ be two (positive) measures. Prove that $\lambda << (\lambda + \mu)$. Also prove that if $\mu \perp (\lambda + \mu)$ then $\mu = 0$.

8 (20pts) Let $(X, \mathbb{X}, \nu)$ be any measure space and let $f \in L_{p_1}$ and $f \in L_{p_2}$ with $1 \leq p_1 < p_2 < \infty$. Prove that $f \in L_p$ for any value of p such that $p_1 \leq p \leq p_2$.

MATH 501-SPRING 2000 **Final Exam**

INSTRUCTOR: Salah Mohammed

NAME: ID #

Answer all questions. Closed book and notes. Show all of your work. Please turn in these sheets with your answers.

1. (25 points) Distinguish between an *algebra* and a *σ-algebra*.

(a) Let $\mathcal{C}$ be the collection of all finite unions of intervals of the form $(a, b]$, or (b, ∞), where $-\infty \leq a < b < \infty$. Is $\mathcal{C}$ an algebra? Is it a σ-algebra? Justify your answer.

(b) Let X, Y be non-empty sets. Suppose $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras of subsets of X and Y respectively. Let $f : X \to Y$ be a mapping. Is $f(\mathcal{F})$ a σ-algebra? Is $f^{-1}(\mathcal{G})$ a σ-algebra? If yes, justify your answer. If no, give counterexamples.

2. (25 points) Define what is meant by *Lebesgue outer measure* λ^* on $\mathbf{R}$. For any subset A of $\mathbf{R}$ and $\alpha, \beta \in \mathbf{R}$, define the set $\alpha A + \beta := \{\alpha x + \beta : x \in A\}$. Prove the following statements

(a) $\lambda^*(\alpha A + \beta) = |\alpha| \lambda^*(A)$.

(b) λ^* is subadditive: for any sequence of sets $A_n, n \geq 1$,

$$\lambda^*(\bigcup_{n\geq 1} A_n) \leq \sum_{n\geq 1} \lambda^*(A_n).$$

(c)If A is Lebesgue measurable, then so is $\alpha A + \beta$.

Prove all statements you make.

3. (35 points) Define what is meant by a *Lebesgue-measurable* set and a *Borel measurable* set in $\mathbf{R}$. Let λ be Lebesgue measure on $\mathbf{R}$. Prove the following statements:

(a) The union of two Lebesgue measurable sets is Lebesgue measurable.

(b) If E is Lebesgue-measurable, then given $\epsilon > 0$, there is an open set U and a closed set F such that $F \subset E \subset U$ and

$$\lambda(U \backslash E) < \epsilon, \quad \lambda(E \backslash F) < \epsilon$$

(c) For every Lebesgue-measurable set E, there is a Borel measurable set B such that $E \subseteq B$ and $\lambda(E) = \lambda(B)$.

4. (25 points) Let $\phi : [0,1] \to [0,1]$ be the Cantor ternary function. Define $f : [0,1] \to \mathbf{R}$ by $f(x) := \frac{1}{2}(\phi(x) + x)$ for all $x \in [0,1]$.

(a) Show that f is a homeomorphism of $[0,1]$ onto itself.

(b) Show that f maps the Cantor set onto a set of Lebesgue measure $\frac{1}{2}$.

(c) Show by example that if $g : [0,1] \to [0,1]$ is continuous and $h : [0,1] \to [0,1]$ is Lebesgue measurable, then $h \circ g : [0,1] \to [0,1]$ may *not* be Lebesgue measurable. Is $h \circ g$ Borel measurable if h is Borel measurable?

5. (35 points) Define what is meant by a *Vitali covering* of a subset $E \subset \mathbf{R}$. State (without proof) the *Vitali Covering Lemma.*

(a) Suppose that the function $f : [a,b] \to \mathbf{R}$ is absolutely continuous on $[a,b]$ and $f'(x) = 0$ for almost every $x \in (a,b)$. Prove that f is constant everywhere on $[a,b]$.

(b) Show that there is a *unique* absolutely continuous $g : [0,1] \to \mathbf{R}$ such that $g(0) = 0$ and $g'(x) = 1$ for almost every x in $(0,1)$. Does uniqueness hold if g is allowed to be of bounded variation on $[0,1]$? If yes, justify your answer; if no, give a counterexample.

6. (25 points) Let $L^p := L^p(0,1), 1 \leq p \leq \infty$, be the set of all Lebesgue-measurable functions $f : (0,1) \to \mathbf{R}$ such that $|f|^p$ is integrable with respect to Lebesgue measure λ on $(0,1)$. Define the associated L^p-norm $\|\cdot\|_p$

(a) Show that the class of all simple functions is dense in L^p.

(b) Prove that $L^q \subseteq L^p$ if $p \leq q$. Show also that the inclusion map $L^q \to L^p$ is continuous linear.

(c) Prove that $L^\infty \subset L^p$ for all $p \geq 1$.

(d) Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$ for each $f \in L^\infty$.

7. (30 points) State *Fatou's Lemma, Lebesgue's Dominated Convergence Theorem* and the *Monotone Convergence Theorem.* Use the Monotone Convergence Theorem to prove Fatou's Lemma.

Let $f_n : \mathbf{R} \to [\mathbf{0}, \infty), \mathbf{n} \geq \mathbf{1}$, be a sequence of non-negative intergrable functions on $\mathbf{R}$ such that $f_n \to f$ almost everywhere, and f is an integrable function over $\mathbf{R}$. Suppose that

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_n(x)\, dx = \int_{\mathbf{R}} f(x)\, dx.$$

Show that for any measurable set E, one has

$$\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx$$

Mathematics 501 Fall 2000

Ph.D. Qualifying Examination

Solve 8 of the following 10 problems. Note that m is Lebesgue measure on the real line $\mathbb{R}$.

1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable for $n = 1, 2, 3, \ldots$. Suppose that

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} |f_n| \, dm < +\infty.$$

Prove that $f_n(x) \to 0$ for almost every $x \in \mathbb{R}$.

2. Let $(X, \mathcal{B}, \lambda)$ be a finite measure space. Suppose that μ is a finite measure on $(X, \mathcal{B})$ that is absolutely continuous with respect to λ; that is, if $A \in \mathcal{B}$ satisfies $\lambda(A) = 0$ then $\mu(A) = 0$. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\mu(A) < \epsilon \quad \text{whenever} \quad A \in \mathcal{B} \quad \text{satisfies } \lambda(A) < \delta.$$

3. Let $\phi_n : [0, 1] \to [0, \infty)$ be monotone increasing for $n = 1, 2, 3, \ldots$. Suppose that the series

$$\sum_{n=1}^{\infty} \phi_n(x) =: f(x)$$

converges for all $x \in [0, 1]$.

a. Prove that f' and

$$\sum_{n=1}^{\infty} \phi_n'$$

exist almost everywhere and are Lebesgue integrable.

b. Prove that

$$f' = \sum_{n=1}^{\infty} \phi_n' \quad \text{a.e.}$$

4. If $J \subset [0,1]$ is an interval let $\mathcal{L}(J)$ denote the length of J. For $A \subset [0,1]$ define

$$\lambda^*(A) = \inf \sum_{k=1}^{N} \mathcal{L}(J_k),$$

where the infimum is taken over all N and all **finite** families of intervals that satisfy

$$A \subset \bigcup_{k=1}^{N} J_k.$$

a. Prove that λ^* is finitely subadditive.

b. Prove that λ^* is **not** countably subadditive.

5. For s a positive real number and A a subset of the reals define

$$H_*^s(A) = \inf \left\{ \sum_{j=1}^{\infty} (b_j - a_j)^s : A \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.$$

a. Prove that H_*^s is an outer measure on $\mathbb{R}$.

b. If $s > 1$ show that $H_*^s(A) = 0$ for every $A \subset \mathbb{R}$.

c. Let C be the Cantor set. Show that $H_*^s(C) = 0$ if $s > \log_3 2$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and Lebesgue measurable. Suppose that there are constants $A > 0$ and $\alpha < 1$ such that

$$m\{x \in \mathbb{R} : |f(x)| > \epsilon\} < \frac{A}{\epsilon^\alpha}$$

for every $\epsilon > 0$. Prove that f is integrable over $\mathbb{R}$.

7. Let $E_k \subset \mathbb{R}$ be a sequence of Lebesgue measurable sets that satisfies

$$\sum_{k=1}^{\infty} m(E_k) < +\infty.$$

Prove that there is a Lebesgue null set N such that every $x \in \mathbb{R} \backslash N$ is in at most a finite number of sets E_k.

8. For $n = 0, 1, 2, 3, \ldots$ and $k = 0, 1, 2, 3, \ldots, 2^n - 1$ define

$$f_{(k+2^n)}(x) = \begin{cases} 1, & \text{if } \frac{k}{2^n} \le x < \frac{k+1}{2^n}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $f_{(k+2^n)}$ is the characteristic (or indicator) function of the interval $[k/2^n, (k+1)/2^n)$.

a. For each $x \in [0, 1)$ compute the following limit or show it does not exist. Justify your answer.

$$\lim_{j \to \infty} f_j(x).$$

b. Let $\phi : [0, 1] \to \mathbb{R}$ be continuous. Compute the following limit or show it does not exist. Justify your answer.

$$\lim_{j \to \infty} \int_{[0,1]} \phi f_j \, dm.$$

c. Does f_j converge in measure? If so find the function it converges to and prove that it converges in measure to that function. If not prove that it does not converge in measure.

For $n = 1, 2, 3, \ldots$ define

$$g_n(x) = \sum_{k=0}^{2^n - 1} 2^{(n-k)} f_{(k+2^n)}(x).$$

d. Compute

$$\int_{[0,1]} g_n \, dm$$

and prove that

$$\lim_{j \to \infty} \int_{[0,1]} g_n \, dm = 2.$$

e. For each $x \in [0, 1)$ compute the following limit or show it does not exist. Justify your answer.

$$\lim_{n \to \infty} g_n(x).$$

f. (Extra Credit) Let $\phi : [0, 1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n \to \infty} \int_{[0,1]} \phi g_n \, dm = 2\phi(0).$$

9. Let f be a real-valued function on the measure space $(X, \mathcal{B})$. For each $t \in \mathbb{R}$, define the set

$$B(t) := \{x \in X : f(x) \leq t\}.$$

If f is measurable then, for each t, $B(t)$ is measurable and

$$(\dagger) \qquad \bigcup_{t\in\mathbb{R}} B(t) = X, \qquad \bigcap_{t\in\mathbb{R}} B(t) = \emptyset, \qquad s < t \implies B(s) \subset B(t).$$

a. Prove that

$$\bigcap_{s<t} B(t) = B(s).$$

b. Suppose instead that $B(t)$, $t \in \mathbb{R}$, is a family of measurable sets that satisfies ($\dagger$) and (a). In this case prove that there exists a unique measurable function $f : X \to \mathbb{R}$ such that

$$B(t) := \{x \in X : f(x) \leq t\} \quad \text{for all } t \in \mathbb{R}.$$

10. Let $f_n \to 0$ a.e. where f_n are real-valued Lebesgue measurable functions on $[0, 1]$. Suppose that there is $K > 0$ such that

$$\int_{[0,1]} |f_n|^2 dm \leq K$$

for every n. Show that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n| \, dm = 0.$$

MATH 501-SPRING 2009 | Final Exam
INSTRUCTOR: Salah Mohammed

NAME:

Answer all questions. Show all of your work. Please turn in this sheet with your answers.

The symbol λ stands for Lebesgue measure on the real numbers $\mathbf{R}$. The symbol $\mathbf{R}^*$ denotes the extended reals.

1. (30 points) Define *Lebesgue outer measure* λ^* on $\mathbf{R}$. Prove the following statements:

   (a) If $A \subset \mathbf{R}$ is bounded, then $\lambda^*(A) < \infty$. Is the converse of this statement true? If yes, prove it; if no, give a counterexample.

   (b) If $E \subset [0,1]$ is a Lebesgue measurable set such that $\lambda(E) = 1$, then E is dense in $[0,1]$.

   (c) Suppose $E \subset (0,1)$ is Lebesgue measurable and $\lambda(E) > 0$. Then, for any $0 < x < \lambda(E)$, there is an open interval $I \subset (0,1)$ such that $\lambda(E \cap I) = x$. (Hint: Consider the function $f(t) := \lambda(E \cap (0,t))$, $t \geq 0$.)

2. (30 points) Define the *Lebesgue integral* for a *bounded measurable function* $f : E \to \mathbf{R}$ over a measurable set E of finite measure. If $g : E \to \mathbf{R}$ is also a bounded measurable function, prove that

$$\int_E (f+g)(x)\,dx = \int_E f(x)\,dx + \int_E g(x)\,dx.$$

3. (30 points) Define what is meant by an extended real-valued *Lebesgue integrable* function $f : \mathbf{R} \to \mathbf{R}^*$.

   If $f : \mathbf{R} \to \mathbf{R}^*$ is Lebesgue integrable and $\epsilon > 0$, prove that there is a measurable set $E \subset \mathbf{R}$ such that $\lambda(E) < \infty$ and $\int_{E^c} |f(x)|\,dx < \epsilon$. ($E^c$ denotes the complement of E in $\mathbf{R}$.)

Define the function $f : (0,\infty) \to \mathbf{R}$ by $f(x) := \dfrac{\sin x}{x}$ for all $x > 0$. Show that f is not Lebesgue integrable over $(0,\infty)$.
(Hint: Consider the series $\sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} (\sin x/x)\, dx$.)

4. (35 points) Define what is meant by *convergence almost everywhere, convergence in measure* and *convergence in mean* for a sequence of measurable functions $f_n : \mathbf{R} \to \mathbf{R},\ n \geq 1$.

Consider the sequence of functions $f_n : \mathbf{R} \to \mathbf{R}$, $n \geq 1$, defined by $f_n(x) := 1_{[n,n+1]}(x)$, $x \in \mathbf{R}$, $n \geq 1$, where $1_{[n,n+1]}$ is the indicator function of the interval $[n, n+1]$. Does the sequence $\{f_n\}_{n=1}^{\infty}$ converge almost everywhere? Does it converge in measure? Does it converge in mean? Justify your answer in each case.

5. (40 points) State *Fatou's Lemma* and the *Monotone Convergence Theorem.* Use Fatou's Lemma to prove the Monotone Convergence Theorem.

Let $f_n : \mathbf{R} \to [0,\infty]$, $n \geq 1$, be a monotone *decreasing* sequence of Lebesgue measurable functions such that f_{n_0} is integrable for some $n_0 \geq 1$. Prove the following statements:

(i) For each $n \geq n_0$, f_n is integrable on $\mathbf{R}$.

(ii) There is an integrable function $f : \mathbf{R} \to [0,\infty]$ such that $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ for a.e. $x \in \mathbf{R}$.

(iii) $\int_{\mathbf{R}} f(x)\, dx = \lim_{n\to\infty} \int_{\mathbf{R}} f_n(x)\, dx$.

Show by example that the requirement that f_{n_0} is integrable for some $n_0 \geq 1$ is necessary for statement (iii) to hold. (Hint: Consider the sequence $f_n := 1_{[n,\infty)}$, $n \geq 1$.)

6. (35 points) Define what is meant by a *Vitali covering* of a subset $E \subset \mathbf{R}$. State (without proof) the *Vitali Covering Lemma.*

Suppose that the function $f : [a,b] \to \mathbf{R}$ is absolutely continuous on $[a,b]$ and $f'(x) = 0$ for almost every $x \in (a,b)$. Prove that f is constant everywhere on $[a,b]$.