580 Qualifying Examination
September 1, 2005

1. Let $X \sim \text{Binomial}(n, p)$ where the positive integer n is large and $0 < p < 1$.

   a) Find the limiting distribution of $\sqrt{n}\left(\frac{X}{n} - p\right)$.

   b) Find the limiting distribution of $\sqrt{n}\left[\left(\frac{X}{n}\right)^2 - p^2\right]$.

   c) Show how to find the limiting distribution of $\left[\left(\frac{X}{n}\right)^3 - \frac{X}{n}\right]$ when $p = \frac{1}{\sqrt{3}}$.

2. Let $X_1, ..., X_n$ be independent identically distributed random variables with probability mass function

$$f(x) = P(X = x) = \frac{1}{x^\nu \zeta(\nu)}$$

   where $\nu > 1$ and $x = 1, 2, 3, ...$ Here the zeta function

$$\zeta(\nu) = \sum_{x=1}^{\infty} \frac{1}{x^\nu}$$

   for $\nu > 1$.

   a) Find a minimal sufficient statistic for ν.

   b) Is the statistic found in a) complete? (prove or disprove)

   c) Give an example of a sufficient statistic that is strictly not minimal.

   d) Consider the family of distributions:

$$\mathcal{P} = \left\{ f(x) = P(X = x) = \frac{1}{x^\nu \zeta(\nu)},\ \nu = 2 \text{ or } 3 \right\}.$$

   Is $\mathcal{P}$ complete? Show why or why not.

3. Let $X_1, ..., X_n$ be independent identically distributed random variables with probability density function

$$f(x) = \frac{\sigma^{1/\lambda}}{\lambda} \exp\left[-(1+\frac{1}{\lambda})\log(x)\right] I[x \geq \sigma]$$

where $x \geq \sigma$, $\sigma > 0$, and $\lambda > 0$. The indicator function $I[x \geq \sigma] = 1$ if $x \geq \sigma$ and 0, otherwise. Find the maximum likelihood estimator (MLE) $(\hat{\sigma}, \hat{\lambda})$ of (σ, λ) with the following steps.

a) Explain why $\hat{\sigma} = X_{(1)} = \min(X_1, ..., X_n)$ is the MLE of σ regardless of the value of $\lambda > 0$.

b) Find the MLE $\hat{\lambda}$ of λ if $\sigma = \hat{\sigma}$ (that is, act as if $\sigma = \hat{\sigma}$ is known).

4. Let $X_1, ..., X_n$ be independent identically distributed random variables with probability density function

$$f(x) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0.$$

a) Find the MLE of $\frac{1}{\theta}$. Is it unbiased? Does it achieve the information inequality lower bound?

b) Find the asymptotic distribution of the MLE of $\frac{1}{\theta}$.

c) Show that $\overline{X}_n$ is unbiased for $\frac{\theta}{\theta+1}$. Does $\overline{X}_n$ achieve the information inequality lower bound?

d) Find an estimator of $\frac{1}{\theta}$ from part (c) above using $\overline{X}_n$ which is different from the MLE in (a). Find the asymptotic distribution of your estimator using the delta method.

e) Find the asymptotic relative efficiency of your estimator in (d) to the MLE in (b).

5. Let X be one observation from the probability density function

$$f(x) = \theta x^{\theta-1}, \;\; 0 < x < 1, \;\;\; \theta > 0.$$

a)Find the most powerful level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

b) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the size and the power function of the test which rejects H_0 if $X > \frac{5}{8}$.

c) Is there a UMP test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, find it. If not, prove so.