Ph.D. Qualifying Examination in Statistics
August, 2006

1. Let $X_n \sim \text{Poisson}(n\lambda)$ where the positive integer n is large and $0 < \lambda$.

   a) Find the limiting distribution of $\sqrt{n}\left(\frac{X_n}{n} - \lambda\right)$.

   b) Find the limiting distribution of $\sqrt{n}\left[\sqrt{\frac{X_n}{n}} - \sqrt{\lambda}\right]$.

2. Let $X_1, ..., X_n$ be independent identically distributed (iid) random variables with probability density function

$$f(x) = \frac{2}{\sqrt{2\pi}\lambda} e^x \exp\left(\frac{-(e^x - 1)^2}{2\lambda^2}\right)$$

   where $x > 0$ and $\lambda > 0$.

   a) Find the maximum likelihood estimator (MLE) $\hat{\lambda}$ of λ.

   b) Find the MLE of λ^2.

3. Let $Y_1, ..., Y_n$ denote a random sample from a uniform $(\theta, \theta + 1)$ population.

   a) Find a two-dimensional sufficient statistic for θ.

   b) Find the method of moments estimator of θ. Name this estimator $\hat{\theta}_1$.

   c) Find an unbiased estimator of θ based on $Y_{(1)} = \min(Y_1, ..., Y_n)$. Name this estimator $\hat{\theta}_2$.

   d) Find the mean square errors of the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$. Which estimator do you choose using the MSE criterion and why?

4. Let $Y_1, ..., Y_n$ be iid random variables with pdf

$$f(y) = \frac{2}{\sqrt{2\pi}\lambda} \frac{1}{y} I_{[0,1]}(y) \exp\left[\frac{-(\log(y))^2}{2\lambda^2}\right]$$

   where $\lambda > 0$. Then $[\log(Y_i)]^2 \sim G(1/2, 2\lambda^2) \sim \lambda^2 \chi^2_1$.

   a) Find the uniformly minimum variance estimator (UMVUE) of λ^2.

   b) Find the information number $I_1(\lambda)$.

   c) Find the Cramér Rao lower bound (CRLB) for estimating $\tau(\lambda) = \lambda^2$.

5. Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\sqrt{2\pi}\lambda} \frac{1}{x} \exp\left[\frac{-(\log(x))^2}{2\lambda^2}\right]$$

where $\lambda > 0$ where and $0 \leq x \leq 1$.

a) What is the UMP (uniformly most powerful) level α test for $H_o : \lambda = 1$ vs. $H_1 : \lambda = 2$?

b) If possible, find the UMP level α test for $H_o : \lambda = 1$ vs. $H_1 : \lambda > 1$.

6. Let $Y_1, ..., Y_n$ denote a random sample from a $N(a\theta, \theta)$ population.

a) Find the MLE of θ when $a = 1$.

b) Find the MLE of θ when a is unknown.

c) Find the likelihood ratio test of $H_0 : a = 1$ versus $H_1 : a \neq 1$ where θ is unknown. Simplify.