Ph.D. Qualifying Examination in Statistics
4:30–8:30 Tuesday, January 23, 2007

1. Let $Y_1, ..., Y_n$ be independent and identically distributed (iid) from a Gamma$(\alpha, \beta)$ distribution.

   a) Find the exact distribution of $\overline{Y}$.

   b) Find the moment generating function of $\sqrt{n}\ \left( \overline{Y} - \ \alpha\beta \right)$.

   c) Find the limiting distribution of $\sqrt{n}\ \left( \overline{Y} - \ \alpha\beta \right)$.

   d) Find the limiting distribution of $\sqrt{n}\ \left( (\overline{Y})^2 - \ c \right)$ for appropriate constant $c$.

2. Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\sqrt{2\pi}\lambda} \frac{1}{x} \exp\left[\frac{-(\log(x))^2}{2\lambda^2}\right]$$

   where $\lambda > 0$ where and $0 \leq x \leq 1$.

   a) Find the maximum likelihood estimator (MLE) of $\lambda$.

   b) Find the MLE of $\lambda^2$.

3. a) State Basu's Theorem.

   b) Let $X_1, ..., X_{10}$ be independent and identically distributed from a Gamma$(\alpha, \beta)$ distribution. Find

$$E\left(\frac{X_3 + X_4 + X_5}{X_1 + \cdots + X_{10}}\right).$$

4. Let $X_1, ..., X_n$ be a random sample from a population with probabity mass function

$$P_\theta(X = x) = \theta^x(1 - \theta)^{1-x}, \quad x = 0 \text{ or } 1, \quad 0 \leq \theta \leq \frac{1}{2}.$$

   a) Find the method of moment estimator of $\theta$. Name this estimator as $\hat{\theta}_1$.

   b) Find the maximum likelihood estimator of $\theta$. Name this estimator as $\hat{\theta}_2$.

   c) Find the mean squared errors of the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$.

   d) Which of the two estimators would be preferred? Justify your choice.

5. Let $X_1, ..., X_n$ be independent identically distributed (iid) random variables with probability density function

$$f(x) = \frac{2}{\sqrt{2\pi}\lambda} e^x \exp\left(\frac{-(e^x - 1)^2}{2\lambda^2}\right)$$

where $x > 0$ and $\lambda > 0$.

a) What is the UMP (uniformly most powerful) level $\alpha$ test for $H_o : \lambda = 1$ vs. $H_1 : \lambda = 2$ ?

b) If possible, find the UMP level $\alpha$ test for $H_o : \lambda = 1$ vs. $H_1 : \lambda > 1$.

6. Let $X_1, ..., X_n$ be a random sample from Poisson($\lambda$). We like to estimate $\theta = P(X = 0) = e^{-\lambda}$.

a) Let $\hat{\theta}_1$ be the maximum likelihood estimator of $\theta$. Find $\hat{\theta}_1$.

b) Let $\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n} Y_i$, where $Y_i = 1$ if $X_i = 0$, and $Y_i = 0$, otherwise. Find the asymptotic distributions of $\hat{\theta}_1$ and $\hat{\theta}_2$.

c) Find the asymptotic relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$.