Ph.D. Qualifying Examination in Statistics
4:30–8:30 Thursday, August 27, 2009

1. Let $X_1, ..., X_n$ be iid uniform$(\theta, \theta + 1)$ random variables where θ is real.

   a) Find a minimal sufficient statistic for θ.

   b) Show whether the minimal sufficient statistic is complete or not.

2. Let $X_1, ..., X_n$ be a random sample from a population with pdf

$$f(x|\mu, \sigma) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, \quad x \geq \mu$$

   where $-\infty < \mu < \infty$, $\sigma > 0$. a) Find the maximum likelihood estimates of μ and σ.

   b) Evaluate $\tau(\mu, \sigma) = P_{\mu,\sigma}[X_1 \geq t]$ where $t > \mu$. Find the maximum likelihood estimator of $\tau(\mu, \sigma)$.

3. Let $X_1, ..., X_n$ be a random sample from $N(\mu_1, \sigma^2)$ population, and independently, let $Y_1, ..., Y_m$ be a random sample from $N(\mu_2, \sigma^2)$ population. Find the UMVU estimators of μ_1, μ_2 and σ^2.

4. Suppose that the test statistic $T(X)$ for testing $H_0 : \lambda = 1$ versus $H_1 : \lambda > 1$ has an exponential$(1/\lambda_1)$ distribution if $\lambda = \lambda_1$. The test rejects H_0 if $T(X) < \log(100/95)$.

   a) Find the power of the test if $\lambda_1 = 1$.

   b) Find the power of the test if $\lambda_1 = 50$.

5. Let $X_1, ..., X_n$ be a random sample from the following gamma p.d.f.

$$\frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\theta}}, \quad 0 < x < \infty,$$

   where $\theta > 0$ is the unknown parameter and α is **known**.

   a) Derive the method of moment estimate of θ.

   b) Let $\hat{\theta}$ be the estimator in (a). Find the asymptotic distribution of $\hat{\theta}$ using the delta method. c) For estimating $\tau(\theta) = \theta$, compute the CRLB for var$(\hat{\theta})$, and compare with the asymptotic variance you calculated in (b) above. Is $(\hat{\theta})$ asymptotically efficient?

6. Let $X_1, ..., X_n$ be independent identically distributed random variables from a Burr type X distribution with pdf

$$f(x) = \ 2\ \tau\ x\ e^{-x^2}\ (1 - e^{-x^2})^{\tau - 1}$$

where $\tau > 0$ and $x > 0$.

a) What is the UMP (uniformly most powerful) level α test for $H_0 : \tau = 2$ versus $H_1 : \tau = 4$?

b) If possible, find the UMP level α test for $H_0 : \tau = 2$ versus $H_1 : \tau > 2$.

7. Let $X_1, ..., X_n$ be a random sample from uniform $(0, \theta)$. Let

$$Y = \max(X_1, ..., X_n).$$

a) Find the pdf of $U = \frac{Y}{\theta}$.

b) To find a confidence interval of θ, can U be used as a pivot?

c) Find the *shortest* confidence interval for θ.