Ph.D. Qualifying Examination in Statistics
4:30–8:30 Thursday, January 22, 2009

1. Suppose that X has probability density function

$$f_X(x) = \frac{\theta}{x^{1+\theta}}, \quad x \geq 1.$$

a) If $U = X^2$, derive the probability density function $f_U(u)$ of U.

b) Find the method of moments estimator of θ.

c) Find the method of moments estimator of θ^2.

2. $X_1, X_2, \ldots X_n$ is a random sample from a normal distribution with mean 5 and unknown variance σ^2.

a) Find the UMVUE of σ^2 (justify your answer using the Rao-Blackwell Theorem and related results).

b) If $n = 1$, find the minimal sufficient statistics (justify).

3. In problem (2) (general n), find the Cramer-Rao lower bound for an unbiased estimator of

a) σ^2,

b) σ.

c) Find an efficient estimator of σ^2.

4. Suppose that $Y_1, ..., Y_n$ are independent binomial(m_i, ρ) where the $m_i \geq 1$ are known constants. Let

$$T_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n m_i} \quad \text{and} \quad \mathrm{T_2} = \frac{1}{\mathrm{n}} \sum_{\mathrm{i=1}}^{\mathrm{n}} \frac{\mathrm{Y_i}}{\mathrm{m_i}}$$

be estimators of ρ.

a) Find MSE(T_1).

b) Find MSE(T_2).

c) Which estimator is better?

Hint: by the arithmetic–geometric–harmonic mean inequality,

$$\frac{1}{n}\sum_{i=1}^n m_i \geq \frac{1}{n}\sum_{i=1}^n \frac{1}{m_i}.$$

5. Suppose that the joint probability distribution function of $X_1, ..., X_k$ is

$$f(x_1, x_2, ..., x_k|\theta) = \frac{n!}{(n-k)!\theta^k} \exp\left(\frac{-[(\sum_{i=1}^k x_i) + (n-k)x_k]}{\theta}\right)$$

where $0 \leq x_1 \leq x_2 \leq \cdots \leq x_k$ and $\theta > 0$.

a) Find the maximum likelihood estimator (MLE) for θ.

b) What is the MLE for θ^2? Explain briefly.

6. Let $X_1, ..., X_n$ be independent identically distributed random variables from a half normal HN(μ, σ^2) distribution with pdf

$$f(x) = \frac{2}{\sigma\ \sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x > \mu$ and μ is real. **Assume that μ is known.**

a) What is the UMP (uniformly most powerful) level α test for
$H_0 : \sigma^2 = 1$ vs. $H_1 : \sigma^2 = 4$?

b) If possible, find the UMP level α test for $H_0 : \sigma^2 = 1$ vs. $H_1 : \sigma^2 > 1$.