Ph.D. Qualifying Examination in Statistics
4:30–8:30 Thursday, January 28, 2010

1. Let $Y_1, ..., Y_n$ be independent and identically distributed (iid) from a distribution with probability mass function $f(y) = \rho(1-\rho)^y$ for $y = 0, 1, 2, ...$ and $0 < \rho < 1$. Then $E(Y) = (1-\rho)/\rho$ and $\text{VAR}(Y) = (1-\rho)/\rho^2$.

   a) Find the limiting distribution of $\sqrt{n}\left(\overline{Y} - \dfrac{1-\rho}{\rho}\right)$.

   b) Show how to find the limiting distribution of $g(\overline{Y}) = \frac{1}{1+\overline{Y}}$. Deduce it completely.

   c) Find the method of moments estimator of ρ.

2. Let $X_1, ..., X_n$ be iid with pdf

$$f(x) = \frac{\cos(\theta)}{2\cosh(\pi x/2)} \exp(\theta x)$$

   where x is real and $|\theta| < \pi/2$.

   a) Find the maximum likelihood estimator (MLE) for θ.

   b) What is the MLE for $\tan(\theta)$? Explain briefly.

3. Let $X_1, ..., X_n$ be iid from a Poisson(θ) distribution. Find the uniformly minimum variance unbiased estimator of $g(\theta) = P_\theta(X = 1) = \theta e^{-\theta}$.

4. Let $X_1, ..., X_n$ be independent identically distributed random variables from an inverse exponential distribution with pdf

$$f(x) = \frac{\theta}{x^2} \exp\left(\frac{-\theta}{x}\right)$$

   where $\theta > 0$ and $x > 0$.

   a) What is the UMP (uniformly most powerful) level α test for
   $H_0 : \theta = 1$ versus $H_1 : \theta = 2$?

   b) If possible, find the UMP level α test for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

5. Let $\theta = (p_1, p_2, \ldots, p_5)$, $p_i \geq 0$, $\sum_{i=1}^{5} p_i = 1$. Suppose $X_1, ..., X_n$ are discrete random variables with $P_\theta(X_i = j) = p_j$, $1 \leq j \leq 5$.

   Consider testing

$$H_0 : p_1 = p_2 = p_3 \text{ versus } H_1 : H_0 \text{ is not true.}$$

   Let y_j = number of $x_1, ..., x_n$ equal to j, $1 \leq j \leq 5$.

   a) Show that the likelihood ratio test statistic can be expressed as

$$-2 \log \lambda(x) = 2 \sum_{i=1}^{5} y_i \log \left(\frac{y_i}{m_i} \right),$$

   where m_i are the expected frequencies. Find expressions for m_i, $1 \leq i \leq 5$.

   b) Find the large sample distribution of the likelihood ratio test statistic.

6. Let $X_1, ..., X_n$ be a random sample from a gamma $(\alpha = 5, \beta)$ population. Consider testing $H_0 : \beta = 7$ versus $H_1 : \beta \neq 7$.

   a) What is the MLE of β?

   b) Derive a Wald statistic for testing H_0, using the MLE in both the numerator and denominator of the statistic.

   c) What is the large sample distribution of the Wald statistic in (b)?