Ph. D. Qualifying Examination in Statistics
Thursday January 19, 2012

1. Let X_1, X_2 be independent random variables. The density function of X_1 is

$$f(t) = \begin{cases} \dfrac{e^t}{e\text{-}1}, & 0 \le t \le 1, \\ 0, & otherwise. \end{cases}$$

The distribution of X_2 is

$$P(X_2 = 1) = p \text{ and } P(X_2 = -1) = 1 - p.$$

Compute the moment generating function of $Y = X_1 X_2$.

2. Let $(X_1, X_2, \cdots, X_n)$ be a random sample from a population with pdf

$$f(x \mid \theta) = \begin{cases} \theta x^{\theta - 1}, & 0 < x < 1; \theta > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a) Show that this density function is in the exponential family, and that $\sum_{i=1}^{n} -\ln(X_i)$ is sufficient for θ.

b) If $W_i = -\ln(X_i)$, show that W_i has an exponential distribution with mean $1/\theta$.

c) Show that $2\theta \sum_{i=1}^{n} W_i$ has a chi-square distribution with $2n$ degrees of freedom.

d) Show that $E\left(\dfrac{1}{2\theta \sum_{i=1}^{n} W_i} \right) = \dfrac{1}{2(n-1)}$.

e) What is the minimum variance unbiased estimator (MVUE)for θ?

3. Let $Y_1, Y_2, ..., Y_k$ be independent, where $Y_i \sim \text{Poisson}(\beta N_i), 1 \le i \le k$. Here β is unknown, and $N_1, N_2, ..., N_k$ are fixed known constants.

a) Use the factorization theorem to obtain a sufficient statistic for β.

b) Find the maximum likelihood estimator (MLE) $\hat{\beta}$ of β.

c) Show that $\hat{\beta}$ is unbiased for β, and derive $\text{var}(\hat{\beta})$.

d) An alternative unbiased estimator is $\beta^* = \sum_{i=1}^{k} \dfrac{Y_i}{kN_i}$. Find $\text{var}(\beta^*)$.

e) Which of the two estimators $(\hat{\beta}$ or $\beta^*)$ is preferred? Discuss.

4. Let $(X_1, X_2, \cdots, X_n)$ be a random sample from an exponential distribution with pdf

$$f(x\,|\,\beta) = \begin{cases} \dfrac{1}{\beta} e^{-x/\beta}, & x > 0 \\ 0, & \text{elsewhere,} \end{cases}$$

where $\beta > 0$.

a)Find the most powerful level α test for $H_0 : \beta = \beta_0$, against $H_1 : \beta = \beta_1$ where $\beta_1 > \beta_0$.

b) Is the above test uniformly most powerful for $H_0 : \beta = \beta_0$, against $H_1 : \beta > \beta_0$? Explain.

5. Let $(X_1, X_2, \cdots, X_n)$ be a random sample from a Poisson distribution with mean θ. We are interested in estimating $g(\theta) = P(X_1 = 0) = e^{-\theta}$. Consider the following two estimators:

$$T_{n,1} = e^{-\overline{X}_n} \text{ and } T_{n,2} = \frac{1}{n}\sum_{i=1}^{n} I(X_i = 0),$$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, and I is an indicator function.

a) Find the asymptotic distribution of $T_{n,1}$ (use delta method).

b) Find the asymptotic distribution of $T_{n,2}$.

c) Which estimator is more efficient in estimating $g(\theta)$ when a large sample size is available? Show your argument.

6. Let $(X_1, X_2, \cdots, X_n)$ be a random sample from $N(\theta, \sigma^2)$, where θ_0 is a specified value of θ, and σ^2 is unknown. We are interested in testing

$$H_0 : \theta = \theta_0, \text{ versus } H_1 : \theta \neq \theta_0.$$

a) Show that the test that rejects H_0 when $|\overline{X} - \theta_0| > t_{n-1,\alpha/2}\sqrt{S^2/n}$ is a test of size α, where $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and $t_{n-1,\alpha/2}$ is the $100(1-\frac{\alpha}{2})$ th percentile from a t-distribution with (n-1) degrees of freedom.

b) Show that the test in part (a) can be derived as a likelihood ratio test.