Ph.D. Qualifying Exam in Topology

January 2006

Instructions: Do as many of the ten problems below as you can. Please use a separate sheet of paper for each problem. If a problem takes more than one page, number the pages. Do not use the back of a page. Good luck!

1. Answer each of the following. (Two points each.)
   (a) What is the *product topology*?
   (b) What is the *quotient topology*?
   (c) State the *Extreme Value Theorem* in its most general form.
   (d) What is a *limit point*?
   (e) State the *Urysohn Lemma*.
2. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X with $\mathcal{T} \subset \mathcal{T}'$. Does $(X, \mathcal{T})$ being connected, imply $(X, \mathcal{T}')$ is? What about the reverse implication? Prove all claims.
3. State and prove the *Uniform Continuity Theorem*.
4. We wish to form a torus by gluing the opposite edges of a square region together. Write this out formally using a quotient map.
5. Prove the *Sequence Lemma*: Let X be a topological space. Let $A \subset X$. If there is a sequence of points in A converging to x, then $x \in \bar{A}$. The converse holds if X is a metric space.
6. Show that $\mathbb{R}^\omega$ in the box topology is not metrizable. Hint: exploit the Sequence Lemma.
7. Prove the continuous image of a compact space is compact. Give an example showing that the continuous image of a closed set need not be closed.
8. State and prove the *Lebesgue Number Lemma*.
9. What is the fundamental group of $S^2 \times S^1$?
10. Let B^2 be the closed unit two-dimensional ball. Prove that there is no retraction from B^2 to S^1.