Ph.D. Qualifying Exam in Topology
August 2008

Instructions: Do as many of the ten problems below as you can. Please use a separate sheet of paper for each problem. If a problem takes more than one page, number the pages. Do not use the back of a page. Good luck!

1. (10 points)
   a) Give the definition of an *order relation* on a set.
   b) Given a set with an order relation define the *order topology*.
   c) Define deformation retraction.
   d) Define local connectedness.
   e) State the Tychonoff Theorem.

2. (10 points) The **Extreme Value Theorem** states that if $f : X \to Y$ is continuous with X compact and Y an ordered set with the order topology then there exists points a and b in X such that $f(a) \leq f(x) \leq f(b)$ for every x in X. Prove this.

3. (10 points) Let X and Y be topological spaces with Y compact. Let $x_0 \in X$ and W be an open subset of $X \times Y$ that contains $x_0 \times Y$. Prove that there is a neighborhood U of x_0 in X such that $U \times Y \subset W$.

4. (10 points) Let X be a Lindelöff space and let $f : X \to Y$ be continuous. Prove that $f(X)$ as a subspace of Y is Lindelöff or give a counter example.

5. (10 points) Prove that the connected components of a topological space are both open and closed.

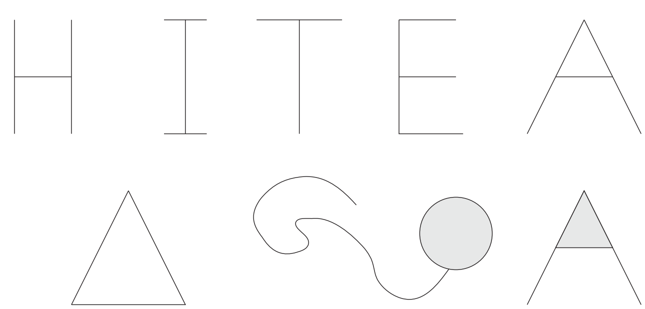
6. (10 points) Prove that a closed subspace of a normal space is normal.

7. (10 points) Consider the product, uniform and box topologies on $\mathbb{R}^\omega$. In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$\begin{aligned} f(t) &= (t, t, t, \ldots) \\ g(t) &= (t, t^2, t^3, \ldots) \\ h(t) &= (\sin t, \sin 2t, \sin 3t, \ldots) \end{aligned}$$

8. (10 points) Let X and Y be topological spaces and let $f : X \to Y$ be a function. Prove that if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$ then f is continuous.

9. (10 points) Classify the figures below up to homoemorphism and by homotopy type. They are to be regarded as subspaces of the plane with the usual topology. Give brief justifications.



10. (10 points) Let A be a deformation retract of X. Let $x_0 \in A$. Prove that the inclusion map $i : (A, x_0) \to (X, x_0)$ induces an isomorphism of fundamental groups.