MATH 108 – REVIEW TOPIC 3

Operations with Polynomials

I. The term polynomial is used to describe any sum of terms provided all coefficients are real and all exponents are positive integers.

Illustration.

Polynomials	Not Polynomials
$3x^4 - 2x^2 + 5x - 11$	$4x^2 - 2x^{-3}$
$\sqrt{2}x^2 - 4x$	$\sqrt{x} + 5$
$x^3y - 2xy^2 + y^3$	$\dfrac{3x^4 - 4}{2x + 5}$

Here are some examples of basic polynomial operations.

Ex. 1: $2x(3x^2 - 4y) = 6x^3 - 8xy^2$

Distributive Property:
$a(b + c) = ab + ac$

Ex. 2:
$$\begin{aligned} & 3x(x^2 - 2x + 5) - (x^3 - 4x - 5) \\ &= 3x^3 - 6x^2 + 15x - x^3 + 4x + 5 \\ &= 2x^3 - 6x^2 + 19x + 5 \end{aligned}$$

Subtraction

Ex. 3:
$$\left.\begin{aligned} & (x - 2)(x^2 + 2x + 4) \\ &= x(x^2 + 2x + 4) - 2(x^2 + 2x - 4) \\ &= x^3 + 2x^2 + 4x - 2x^2 - 4x + 8 \\ &= x^3 + 8 \end{aligned}\right\}$$

Similar to Ex. 1,
Distribute x and
-2

Note: Review Topic 4 deals with factoring, the "undoing" of multiplication. Because $(x - 2)$ and $(x^2 + 2x + 4)$ multiply to become $x^3 + 8$, $x^3 + 8$ factors into $(x - 2)(x^2 + 2x + 4)$. Understanding multiplication patterns is the first step in learning to factor.

II. Long Division

Dividing two polynomials is very much like dividing two integers. Compare the following:

Divide 157 by 11

$$\begin{array}{r|l} & 14 \\ \hline 11 & 157 \\ & \underline{11} \\ & 47 \\ & \underline{44} \\ & 3 \end{array}$$

Ans: $14 + \dfrac{3}{11} = 14\dfrac{3}{11}$

Divide $2x^2 - 5x - 7$ by $x - 3$

$$\begin{array}{r|r} & 2x + 1 \\ \hline x - 3 & 2x^2 - 5x - 7 \\ & \underline{2x^2 - 6x\phantom{{}-7}} \\ & x - 7 \\ & \underline{x - 3} \\ & -4 \end{array}$$

Ans: $2x + 1 + \dfrac{-4}{x - 3}$

Do you see any real difference in these? Hopefully not.

Exercise 1: Divide $x^4 - 2x + 5$ by $x + 2$.

If the missing terms in the dividend create alignment problems, blanks may be helpful.

$$x + 2 \overline{\big)\, x^4 + \underline{\quad} x^3 + \underline{\quad} x^2 - 2x + 5}$$

Answer

III. Products of Binomials

We'll start with 3 examples. Pay close attention to "outer and inner" products and how they determine the middle term of the result.

Ex. 1: $(2x - 3)(x + 5) = 2x^2 + (10x - 3x) - 15 = 2x^2 + 7x - 15$

Ex. 2: $(2x - 3)^2 = (2x)^2 + \overset{\text{twice } -6x}{(-6x - 6x)} + (-3)^2 = 4x^2 - 12x + 9$

Ex. 3: $(2x - 3)(2x + 3) = 4x^2 + (6x - 6x) - 9 = 4x^2 - 9$

Here are a few comments:

Ex. 1. The final result is a trinomial. Summing the outer and inner products yields the middle term.

Ex. 2. Squaring a binomial also yields 3 terms. Identical outer and inner products create a middle term twice as big.

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

Common error:

$$(3x)^2 = 9x^2 \quad \text{but} \quad (3+x)^2 \neq 9 + x^2$$

Remember $(3+x)^2 = 3^2 + \text{twice } 3x + x^2 = 9 + 6x + x^2$.

Ex. 3. Binomials of the type $(a+b)$ and $(a-b)$ are referred to as conjugate pairs. The product of these pairs must be a binomial (no middle term) since the sum of the outers and inners is 0.

$$\underbrace{(a+\overbrace{b)(a}^{ab}-b)}_{-ab} = a^2 - b^2$$

Comments: Once you have thought this through (particularly what determines the middle term) you will be able to write down answers WITHOUT any additional steps.

Exercise 2: Find the following products by inspection (no steps).

a) $(3x-4)(2x+7)$

b) $(3x^2+4)(2x^2-7)$

c) $(2x+7y)(2x-7y)$

(d) $(2x-7y)^2$

(e) $(x^3-1)^2$

(f) $-(2x-5)^2$

Answers

IV. Polynomials – Mixed Operations

Assessment: Perform the indicated operations.

a) $(3x-4)(x+5)-(2x-3)^2$

b) $(3x-4)^3$

c) $(x-2)^2(x+2)^2$

Answers

PRACTICE PROBLEMS for Topic 3

Perform the indicated operation.

3.1. $(2x-3)(x+4)(x-4)$

3.2. $6x^5+3x^4-4x^3-2x^2+2x+4$ divided by $2x+1$

3.3. $(x-y-z)(x-y+z)$

Answers

ANSWERS to PRACTICE PROBLEMS (Topic 3–Operations with Polynomials)

3.1. $(2x-3)(x^2-16) = 2x^3-3x^2-32x+48$
You get the same result from finding the product of $(2x-3)(x+4)$ and then multiplying by $(x-4)$. Multiplication is both commutative and associative:

$$abc = a(bc) = (ab)c = (ac)b\ldots$$

Thus any arrangement of factors gives the same result.

3.2. $3x^4-2x^2+1+\dfrac{3}{2x+1}$
Can you check? Is $(2x+1)(3x^4-2x^2+1)+3 = 6x^5+3x^4-4x^3-2x^2+2x+4$?

3.3. Looking for a shortcut? Group into $[(x-y)-z][(x-y)+z]$. Then let $a = x-y$ and $b = z$ and multiply as conjugates.

$$\begin{aligned}(a-b)(a+b) = a^2-b^2 \Rightarrow [(x-y)-z][(x-y)+z] \\ = [(x-y)^2-z^2] = x^2-2xy+y^2-z^2\end{aligned}$$

Beginning of Topic 108 Skills Assessment

Divide $x^4 - 2x + 5$ by $x + 2$.

Answer:

$$x^3 - 2x^2 + 4x - 10 + \frac{25}{x+2}$$

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Find the following products by inspection (no steps).

a) $(3x-4)(2x+7)$

b) $(3x^2+4)(2x^2-7)$

c) $(2x+7y)(2x-7y)$

(d) $(2x-7y)^2$

(e) $(x^3-1)^2$

(f) $-(2x-5)^2$

Answers

a) $6x^2+13x-28$

b) $6x^4-13x^2-28$

c) $(2x)^2-(7y)^2=4x^2-49y^2$

d) $(2x)^2 - \text{twice } (2x)(7y)+(7y)^2=4x^2-28xy+49y^2$

e) x^6-2x^3+1

f) $-(4x^2-20x+25)=-4x^2+20x-25$
Negation comes after squaring.

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Perform the indicated operations.

a) $(3x-4)(x+5)-(2x-3)^2$ (b) $(3x-4)^3$ (c) $(x-2)^2(x+2)^2$

Answers

a) $3x^2+11x-20-(4x^2-12x+9)$
$= -x^2+23x-29$

Subtraction follows squaring.

b) Common error $(3x-4y)^3 \neq (3x)^3-(4y)^3$

$$(3x-4y)^3 = \underbrace{(9x^2-24xy+16y^2)}_{(3x-4y)^2}(3x-4y)$$

$$= 27x^3-108x^2y+144xy^2-64y^3$$

<u>Alternate Method</u>: If you have experience with binomial expansion (integral powers of a binomial), you may know

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$

Then let $a=3x$ and $b=-4y$ and SUBSTITUTE.

Ans: $(3x-4y)^3 = (3x)^3+3(3x)^2(-4y)+3(3x)(-4y)^2+(-4y)^3$
$= 27x^3-108x^2y+144xy^2-64y^3$.

If you are intrigued by this and want to explore it further, see Sec. 13.5 in the current text, *Algebra & Trigonometry* by M. Sullivan.

c) $(x-2)^2(x+2)^2 = [(x-2)(x+2)]^2 = (x^2-4)^2 = x^4-8x^2+16$.

Isn't this better than squaring each binomial and finding the product of (x^2-4x+4) and (x^2+4x+4).

Look for shortcuts; there are rules to follow but half the fun is learning how to "beat them".

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