MATH 108 – REVIEW TOPIC 5

Rational Expressions

Your instructor would define a rational expression as "the quotient of two polynomials." To you it is the source of that recurring migrane you get whenever you work with fractions.

To be sucessful with rational expressions you must be proficient in fractional arithmetic:

$$\frac{3}{4}+\frac{6}{5}=\,? \qquad \frac{3}{4}\left(-\frac{6}{5}\right)=\,? \qquad \frac{\frac{3}{4}-\frac{6}{5}}{2}=\,?$$

Similar problems involving rational expressions follow the same principles, only with algebra concepts intermixed. Do you fill that headache coming on!?

I. Simplifying

Simplifying a rational expression requires finding and removing **common factors** that appear in both numerator and denominator. Formally you are using the fact that $1\cdot(\quad)=(\quad)$. Informally you know it as "cancelling". For help with factoring, see Review Topic 4.

Below are three fractions, all reduced to lowest terms.

Example: $$\frac{12x^2y}{8xy^3}=\frac{4xy\cdot 3x}{4xy\cdot 2y^2}=\left(\overbrace{\frac{4xy}{4xy}}^{1}\right)\cdot\frac{3x}{2y^2}=\frac{3x}{2y^2}$$

Example: $$\frac{4x^2}{2x(4x-5)}=\left(\overbrace{\frac{2x}{2x}}^{1}\right)\cdot\frac{2x}{4x-5}=\frac{2x}{4x-5}$$

Example: $$\frac{x^2-3x-4}{16-x^2}=\frac{(x-4)(x+1)}{(4-x)(4+x)}=\left(\overbrace{\frac{x-4}{4-x}}^{-1}\right)\cdot\frac{x+1}{4+x}=-\frac{x+1}{4+x}$$

Your own steps may differ from those shown, but there must be agreement on the basic concept: simplifying is the removing of factors (not terms) that are equivalent to 1.

Common Errors:

$$\frac{x+4}{2x} \neq \frac{1+4}{2} \qquad x \text{ is a term, not a factor}$$

$$\frac{x^2-2x}{x^2+5} \neq \frac{-2x}{5} \qquad x^2 \text{ is a term, not a factor.}$$

Repeat after me 10 times: **"Factors are cancelled, not terms."**

Exercise 1. Simplify each expression.

a) $\dfrac{4x^2(x-3)^2}{10x(3-x)}$

b) $\dfrac{x^2-2x}{x^2-5x+6}$

c) $\dfrac{2x^2+7x+3}{2x^2-7x-4}$

d) $\dfrac{y^2-25}{y^3-15}$

Answers

II. Multiplication and Division

There are four things to remember when multiplying or dividing fractions.

1) All fractions can be multiplied (common denominators are only necessary when adding.)
2) The product of two fractions is the product of the numerators over the product of the denominators. $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$.
3) To ensure your answer is in lowest terms, cancel common factors **before multiplying**.
4) Division is defined in terms of multiplication. $a \div b = \dfrac{a}{b} = a\left(\dfrac{1}{b}\right)$

As in simplifying, most of your work comes from factoring. The rest should be easy.

Example:

$$\frac{x^2-2x}{x^2-5x-6}\cdot\frac{2x^2-x-3}{4x^2}$$

$$=\frac{x(x-2)}{(x-6)(x+1)}\cdot\frac{(2x-3)(x+1)}{4x^2}=\left(\overbrace{\frac{x(x+1)}{x(x+1)}}^{1}\right)\cdot\frac{(x-2)(2x-3)}{4x(x-6)}$$

$$=\frac{(x-2)(2x-3)}{4x(x-6)}$$

You probably were taught to cancel equal factors. Repeating the same example:

$$\frac{\cancel{x}(x-2)}{(x-6)\cancel{(x+1)}}\cdot\frac{(2x-3)\cancel{(x+1)}}{4x^{\cancel{2}}}=\frac{(x-2)(2x-3)}{4x(x-6)}$$

Example: Simplify $\dfrac{x+2}{2x-3}\div\dfrac{x^2-4}{3x-2x^2}$;
(Remember our comment about division.)

Answer:

$$\frac{x+2}{2x-3}\div\frac{x^2-4}{3x-2x^2}=\frac{x+2}{2x-3}\cdot\frac{3x-2x^2}{x^2-4}$$

$$=\frac{x+2}{2x-3}\cdot\frac{x(3-2x)}{(x-2)(x+2)}=\left(\frac{x+2}{x+2}\right)\left(\frac{\overbrace{3-2x}^{-1}}{2x-3}\right)\cdot\frac{x}{x-2}$$

$$=-\frac{x}{x-2}\text{ or }\frac{x}{2-x}$$

Exercise 2: Perform the indicated operations.

a) $\dfrac{6x^2-5x-6}{x^2-4}\div\dfrac{2x^2-3x}{x+2}$

b) $\dfrac{4x^2-9}{2x^2+7x+6}\cdot\dfrac{4x^4+6x^3+9x^2}{8x^7-27x^4}$

c) $\dfrac{x^3-2x^2+2x-4}{x^2-2x}\cdot\dfrac{3x^4}{x^4+4x^2+4}$

Answers

III. Addition

Illustration: Compare the following additions; notice the similarity in steps.

1.

$\dfrac{7}{12} - \dfrac{5}{12}$	$\dfrac{6}{x^2-4} - \dfrac{3x}{x^2-4}$
$= \dfrac{2}{12}$	$= \dfrac{6-3x}{x^2-4}$
$= \left(\dfrac{2}{2}\right)\dfrac{1}{6}$	$= \dfrac{3(2-x)}{(x-2)(x+2)}$
$= \dfrac{1}{6}$	$= -\dfrac{3}{x+2}$

2.

$3 - \dfrac{2}{5} - \dfrac{4}{3}$ LCD $= 15$	$2 + \dfrac{3x+1}{x} - \dfrac{x-2}{x^2}$ LCD $= x^2$
$= \dfrac{3}{1}\left(\dfrac{15}{15}\right) - \dfrac{2}{5}\left(\dfrac{3}{3}\right) - \dfrac{4}{3}\left(\dfrac{5}{5}\right)$	$= \dfrac{2}{1}\left(\dfrac{x^2}{x^2}\right) + \dfrac{3x+1}{x}\left(\dfrac{x}{x}\right) - \dfrac{(x-2)}{x^2}$

(Multiplying by a form of 1 gives fractions a common denominator and still maintains equality.)

$= \dfrac{45-6-20}{15}$	$= \dfrac{2x^2+3x^2+x-x+2}{x^2}$
$= \dfrac{19}{15}$	$= \dfrac{5x^2+2}{x^2}$

With no common factors to cancel, answers are in lowest terms.

Common Error: Rational expressions oftentimes have polynomials for numerators. To subtract you must "negate" each term in the numerator.

$$-\frac{x-2}{x^2} = \frac{-(x-2)}{x^2} = \frac{-x+2}{x^2}.$$

Again, we want to continue to stress that all fractions follow the same set of rules.

Guidelines for Adding Rational Expressions

1) Express denominators in factored form.
2) Find the Least Common Denominator (LCD).
3) Convert all fractions into the common denominator by multiplying by a form of 1.
4) Add.
5) Simplify, i.e. search for common factors to cancel.

Example: Simplify the expression $\dfrac{6}{3x^2-2x}+\dfrac{5}{3x-2}-\dfrac{2}{x^2}$.

Answer:

$$\frac{6}{x(3x-2)}+\frac{5}{3x-2}-\frac{2}{x^2} \qquad \text{LCD} = x^2(3x-2)$$

$$=\frac{6x}{x^2(3x-2)}+\frac{5x^2}{x^2(3x-2)}-\frac{2(3x-2)}{x^2(3x-2)}$$

$$=\frac{6x+5x^2-6x+4}{x^2(3x-2)}$$

$$=\frac{5x^2+4}{x^2(3x-2)}$$

No common factors guarantees answer in lowest terms.

Exercise 3: If the following represent denominators, find the LCD.

a) $x(x-3)$ and $x^2(x+3)$

b) $6-2x$ and x^2-5x+6

c) $(x-1)^2$; x^3-x and x^2

Answers

Find each sum.

d) $\dfrac{4x}{3x-4}+\dfrac{8}{3x^2-4x}+\dfrac{2}{x}$

e) $4+\dfrac{2}{u}-\dfrac{3u}{u+5}$

f) $\dfrac{2x+1}{x^2+4x+4}-\dfrac{6x}{x^2-4}+\dfrac{3}{x-2}$

Answers

IV. Complex Fractions

You are probably thinking, "Aren't all fractions complex?" Yes, but as you will soon see, some are MORE complex.

A complex fraction contains fractions in its numerator and/or denominator. Here are two methods you could use to simplify a complex fraction.

Example: a) $\dfrac{\frac{5}{6}-\frac{5}{12}}{2}$ (b) $\dfrac{\frac{1}{x}-\frac{1}{y}}{xy}$

Method 1: Simplify any sum, then divide.

$$=\frac{\frac{10}{12}-\frac{5}{12}}{2} \qquad =\frac{\frac{y}{xy}-\frac{x}{xy}}{xy}$$

$$=\frac{\frac{5}{12}}{2} \qquad =\frac{\frac{y-x}{xy}}{xy}$$

$$=\frac{5}{12}\cdot\frac{1}{2} \qquad =\frac{y-x}{xy}\cdot\frac{1}{xy} \quad \text{Dividing Step}$$

$$=\frac{5}{24} \qquad =\frac{y-x}{x^2y^2}$$

Method 2: Multiply by a form of 1 and cancel.

$$=\left(\frac{\frac{5}{6}-\frac{5}{12}}{2}\right)\frac{12}{12} \qquad =\left(\frac{\frac{1}{x}-\frac{1}{y}}{xy}\right)\frac{xy}{xy}$$

$$=\frac{{}^{2}\cancel{12}\left(\frac{5}{\cancel{6}}\right)-\cancel{12}\left(\frac{5}{\cancel{12}}\right)}{24} \qquad =\frac{(\cancel{x}y)\frac{1}{\cancel{x}}-(x\cancel{y})\frac{1}{\cancel{y}}}{x^2y^2}$$

$$=\frac{10-5}{24} \qquad =\frac{y-x}{x^2y^2}$$

$$=\frac{5}{24}$$

Exercise 4: Simplify $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$; use both methods. Answer

Exercise 5: Simplify

a) $\dfrac{\dfrac{y}{x}-\dfrac{x}{y}}{x-y}$

(c) $\dfrac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$

b) $\dfrac{y^{-1}-x^{-1}}{xy}$ Hint: $y^{-1}-x^{-1}=\dfrac{1}{y}-\dfrac{1}{x}$ Answers

Beginning of Topic 108 Skills Assessment

Simplify each expression.

a) $$\frac{4x^2(x-3)^2}{10x(3-x)}$$

b) $$\frac{x^2-2x}{x^2-5x+6}$$

c) $$\frac{2x^2+7x+3}{2x^2-7x-4}$$

d) $$\frac{y^2-25}{y^3-125}$$

Answers:

1a) $$-\frac{2x(x-3)}{5}$$

1b) $$\frac{x(x-2)}{(x-2)(x-3)} = \frac{x}{x-3}$$

1c) $$\frac{(2x+1)(x+3)}{(2x+1)(x-4)} = \frac{x+3}{x-4}$$

1d) $$\frac{(y-5)(y+5)}{(y-5)(y^2+5y+25)} = \frac{y+5}{y^2+5y+25}$$

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Perform the indicated operations.

a) $\dfrac{6x^2-5x-6}{x^2-4} \div \dfrac{2x^2-3x}{x+2}$

b) $\dfrac{4x^2-9}{2x^2+7x+6} \cdot \dfrac{4x^4+6x^3+9x^2}{8x^7-27x^4}$

c) $\dfrac{x^3-2x^2+2x-4}{x^2-2x} \cdot \dfrac{3x^4}{x^4+4x^2+4}$

Answers:

a) $\dfrac{(3x+2)\cancel{(2x-3)}}{(x-2)\cancel{(x+2)}} \cdot \dfrac{\cancel{x+2}}{x\cancel{(2x-3)}} = \dfrac{3x+2}{x(x-2)}$

b) $\dfrac{(2x-3)(2x+3)}{(2x+3)(x+2)} \cdot \dfrac{x^2(4x^2+6x+9)}{x^4(\underbrace{8x^3-27}_{(2x-3)(4x^2+6x+9)})} = \dfrac{1}{x^2(x+2)}$

c) Using grouping,

$$\begin{aligned} x^3-2x^2+2x-4 &= x^2(x-2)+2(x-2) \\ &= (x^2+2)(x-2) \end{aligned}$$

$$\begin{aligned} \frac{x^3-2x^2+2x-4}{x^2-2x} \cdot \frac{3x^4}{x^4+4x^2+4} &= \frac{(x^2+2)(x-2)}{x(x-2)} \cdot \frac{3x^4}{(x^2+2)^2} \\ &= \frac{3x^3}{x^2+2} \end{aligned}$$

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If the following represent denominators, find the LCD.

a) $x(x-3)$ and $\mathrm{x}^2(x+3)$

b) $6-2x$ and x^2-5x+6

c) $(x-1)^2$; x^3-x and x^2

Answers:

a) $x^2(x-3)(x+3)$

b) $2(x-3)(x-2)$ Note: $3-x=-(x-3)$

c) Since $x^3-x=x(x-1)(x+1)$, LCD $=x^2(x-1)^2(x+1)$

(d), e), and f) are continued on next page.)

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If the following represent denominators, find the LCD.

d) $\dfrac{4x}{3x-4} + \dfrac{8}{3x^2-4x} + \dfrac{2}{x}$

f) $\dfrac{2x+1}{x^2+4x+4} - \dfrac{6x}{x^2-4} + \dfrac{3}{x-2}$

e) $4 + \dfrac{2}{u} - \dfrac{3u}{u+5}$

Answers:

d) LCD $= x(3x-4)$

$$\frac{4x^2+8+2(3x-4)}{x(3x-4)}$$

$$=\frac{4x^2+6x}{x(3x-4)} = \frac{2x(2x+3)}{x(3x-4)} = \frac{2(2x+3)}{3x-4}$$

e) LCD $= u(u+5)$

$$\frac{4u(u+5)+2(u+5)-3u(u)}{u(u+5)}$$

$$=\frac{u^2+22u+10}{u(u+5)}$$

f) LCD $= (x+2)^2(x-2)$

$$\frac{2x+1}{(x+2)^2} - \frac{6x}{(x-2)(x+2)} + \frac{3}{x-2}$$

$$= \frac{(2x+1)(x-2)-6x(x+2)+3(x+2)^2}{(x+2)^2(x-2)}$$

$$= \frac{2x^2-3x-2-6x^2-12x+3x^2+12x+12}{(x+2)^2(x-2)}$$

$$= \frac{-x^2-3x+10}{(x+2)^2(x-2)}$$

$$= \frac{-(x+5)(x-2)}{(x+2)^2(x-2)}$$

$$= \frac{-(x+5)}{(x+2)^2}$$

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Simplify $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$; use both methods.

Answer:

Method 1:

$$\frac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}=\overbrace{\frac{\dfrac{x-(x+h)}{(x+h)x}}{h}=\frac{\dfrac{-h}{(x+h)x}}{h}}^{\text{Add}}=\overbrace{\frac{-h}{(x+h)x}\cdot\frac{1}{h}=\frac{-1}{(x+h)x}}^{\text{Divide}}$$

Method 2: If we multiply by 1,

$$\left(\frac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}\right)\frac{x(x+h)}{x(x+h)}=\frac{x-(x+h)}{hx(x+h)}$$

$$=\frac{-h}{hx(x+h)}$$

$$=\frac{-1}{x(x+h)}$$

Leaving denominator in factored form allows for cancellation.

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Simplify

a) $\dfrac{\dfrac{y}{x} - \dfrac{x}{y}}{x - y}$ (b) $\dfrac{y^{-1} - x^{-1}}{xy}$ (c) $\dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$

Answers:

a) $$\frac{y^2 - x^2}{xy} \cdot \frac{1}{x - y} = \frac{(y - x)(y + x)}{(x - y)xy} = -\frac{y + x}{xy}$$

b) $$\frac{\dfrac{1}{y} - \dfrac{1}{x}}{xy} = \frac{\dfrac{1}{y} - \dfrac{1}{x}}{xy} \cdot \frac{xy}{xy} = \frac{x - y}{x^2y^2}$$

c) $$\frac{\dfrac{x^2 - (x+h)^2}{x^2(x+h)^2}}{h} = \frac{\dfrac{-2xh - h^2}{x^2(x+h)^2}}{h} = \frac{h(-2x - h)}{x^2(x+h)^2} \cdot \frac{1}{h} = \frac{-2x - h}{x^2(x+h)^2} = -\frac{2x + h}{x^2(x+h)^2}$$

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