MATH 108 – REVIEW TOPIC 6

Radicals

The arithmetic and algebra of radicals play a significant role in mathematics. Radicals are used extensively in trigonometry. Since your instructors will insist on exact answers in simplest form (not a decimal approximation), it is important that you master basic manipulations involving radicals.

I. Computations with Radicals

Let's start with a brief assessment.

Exercise 1. Express in simplest form.

a) $\sqrt{16}$

b) $\sqrt[4]{16}$

c) $\sqrt[3]{27}$

d) $\sqrt{27}$

e) $\sqrt{-40}$

f) $\sqrt[3]{-40}$

g) $3\sqrt{5}-\sqrt{5}$

h) $(3\sqrt{5})^2$

i) $(\sqrt{3}-2)(\sqrt{3}+2)$

j) $\sqrt{\dfrac{20}{9}}$

Answers

Comments:

1. Some radicals have an exact root.

$$\sqrt[n]{a}=b \;\text{ if }\; b^n=a.$$

In other words,

$$\begin{aligned} \sqrt{16}&=4 &\quad\text{because}\quad& 4^2=16\\ \sqrt[4]{16}&=2 &\quad\text{because}\quad& 2^4=16\\ \sqrt[3]{64}&=4 &\quad\text{because}\quad& 4^3=64 \end{aligned}$$

Question: If 4^2 and $(-4)^2$ both equal 16, why is $\sqrt{16}$ not written as ± 4?

Answer: $\sqrt{16}=4$ but not -4 because we want to avoid ambiguity in simplifying radicals. Only the positive or "principle" root is written. However, in solving equations, **all possible roots** that satisfy a given equation must be written.

Illustration:

$$x^2=16 \Rightarrow x=\pm\sqrt{16}=\pm 4.$$

2. **Simplifying** refers to factoring the quantity inside a radical and then removing any factor that has an exact root. We'll illustrate simplifying with two slightly different methods:

   a) Perfect Square Factors

      i) $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4} \cdot \sqrt{10} = 2\sqrt{10}$ $\qquad \sqrt{ab} = \sqrt{a}\sqrt{b}$

      ii) $\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}$

      iii) $\sqrt[3]{40} = \sqrt[3]{8 \cdot 5} = \underbrace{\sqrt[3]{8}}_{\text{perfect cube}} \cdot \sqrt[3]{5} = 2\sqrt[3]{5}$

   b) Repeated Factors

      i) $\sqrt{40} = \sqrt{2^2 \cdot 10} = \sqrt{2^2} \cdot \sqrt{10} = 2\sqrt{10}$

      ii) $\sqrt{75} = \sqrt{5^2 \cdot 3} = \sqrt{5^2}\sqrt{3} = 5\sqrt{3}$

      iii) $\sqrt[3]{40} = \sqrt[3]{2^3 \cdot 5} = 2\sqrt[3]{5}$

      for $a > 0$, $\sqrt[n]{a^n} = a$

3. Basic algebra concepts still apply.

   a) Combining Like Terms

      $3\sqrt{5} + 5\sqrt{5} = 8\sqrt{5}$; similar to $3x^2 + 5x^2 = 8x^2$

   b) Exponent Rules

      $(2\sqrt{5})^2 = 4(\sqrt{5})^2 = 4(5) = 20$; similar to $(2a^3)^2 = 4a^6$

      $\sqrt{\dfrac{100}{81}} = \dfrac{\sqrt{100}}{\sqrt{81}} = \dfrac{10}{9}$; similar to $\left(\dfrac{x}{y^2}\right)^3 = \dfrac{x^3}{y^6}$.

   c) Finding Products

      $$\begin{aligned}(2+\sqrt{3})^2 &= 4 + 2(2\sqrt{3}) + (\sqrt{3})^2; \quad \text{similar to } (2x+3)^2 = 4x^2 + 12x + 9 \\ &= 7 + 4\sqrt{3}\end{aligned}$$

      Recall: $(a+b)^2 = a^2 + 2ab + b^2$

      See Review Topic 3 if you need help squaring a binomial by inspection.

II. Radicals Containing Variables

Removing variables from inside a radical is essentially the same as removing constants.

a) $\sqrt{x^2y^3} = \sqrt{(xy)^2y} = xy\sqrt{y}$

b) $\sqrt[3]{x^6} = \sqrt[3]{(x^2)^3} = x^2$ Remember $\sqrt[n]{(\ \ )^n} = (\ \ )$

Alternate Method: In part IV of this Review Topic we discuss $\sqrt[n]{x^m} = x^{m/n}$. Thus $\sqrt[3]{x^6} = x^{6/3} = x^2$.

c) $\sqrt{x^2 + 6x + 9} = \sqrt{(x+3)^2} = x + 3$

WARNING!!! $\sqrt{x^2 + 9} \neq x + 3$

Only repeated factors can be removed from a radical. The reason $\sqrt{9x^2} = 3x$ is because 9 and x^2 are factors. This error is so common it is worth repeating.

$$\sqrt{\mathbf{a^2b^2}} = \mathbf{ab} \ \ \textbf{but} \ \ \sqrt{\mathbf{a^2 + b^2}} \neq \mathbf{a + b}. \ \ \textbf{Only} \ \ \sqrt{\mathbf{(a+b)^2}} = \mathbf{a + b}.$$

d) $\sqrt[3]{\dfrac{3xy^5}{81x^4y^2}} = \sqrt[3]{\dfrac{y^3}{27x^3}} = \dfrac{\sqrt[3]{y^3}}{\sqrt[3]{27x^3}} = \dfrac{y}{3x}$

Notice the fraction was simplified **before** roots were taken.

Exercise 2. Express in simplest form.

a) $\sqrt{x^{16}}$

b) $\sqrt[3]{x^9}$

c) $\sqrt{x^9}$

d) $\sqrt{54x^2y^3}$

e) $\sqrt[3]{54x^2y^3}$

f) $(\sqrt{x} - \sqrt{y})^2$

g) $\sqrt{\left(\dfrac{x}{2}\right)^2 - \left(\dfrac{\sqrt{3}x}{4}\right)^2}$

h) $\sqrt[3]{\dfrac{3x^9y^{10}}{24y^{-2}}}$

i) $\sqrt{4x^2 - 20x + 25}$

j) $\sqrt{4x^2 - 25}$

k) True or False: is $\sqrt{x^2} = x$ for all values of x?

Answers

III. Rationalizing

A fraction containing a radical in its denominator is **NOT** considered to be in simplest form. Removing radicals from the denominator is called "rationalizing".

Illustration:

$$\frac{4}{\sqrt{2}},\quad \sqrt{\frac{1}{3}},\quad \frac{5}{\sqrt[3]{x}},\quad \text{and} \quad \frac{2}{3-\sqrt{2}} \quad \text{can all be simplified by rationalizing.}$$

Here is how it is done.

Rationalizing

a) $\dfrac{4}{\sqrt{2}} = \dfrac{4}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{4\sqrt{2}}{\sqrt{2^2}} = \dfrac{4\sqrt{2}}{2} = 2\sqrt{2}$

b) $\sqrt{\dfrac{1}{3}} = \dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ or $\dfrac{1}{3}\sqrt{3}$

c) $\dfrac{5}{\sqrt[3]{x}} = \dfrac{5}{\sqrt[3]{x}} \cdot \dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \dfrac{5\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \dfrac{5\sqrt[3]{x^2}}{x}$

Have you noticed:

i) each fraction is being multiplied by a form of 1 ...

ii) resulting in a denominator of the form $\sqrt[n]{(\quad)^n}$, thereby causing the radical to be removed.

Rationalizing binomial denominators requires a different approach. Because $(a-b)(a+b) = a^2 - b^2$, multiplying by the appropriate binomial factor (still in the form of 1) will remove radicals.

d) $\dfrac{2}{\sqrt{3}-2} = \dfrac{2}{\sqrt{3}-2} \cdot \dfrac{\sqrt{3}+2}{\sqrt{3}+2} = \dfrac{2(\sqrt{3}+2)}{(\sqrt{3})^2 - 2^2} = \dfrac{2(\sqrt{3}+2)}{3-4} = -2(\sqrt{3}+2)$

e) $\dfrac{3-x}{\sqrt{3}+\sqrt{x}} = \dfrac{3-x}{\sqrt{3}+\sqrt{x}} \cdot \dfrac{\sqrt{3}-\sqrt{x}}{\sqrt{3}-\sqrt{x}} = \dfrac{(3-x)(\sqrt{3}-\sqrt{x})}{3-x} = \sqrt{3}-\sqrt{x}$

Exercise 3. Rationalize; express in simplest form.

a) $\dfrac{4}{\sqrt{6}}$

b) $\sqrt{\dfrac{3}{2x}}$

c) $\dfrac{1}{\sqrt[4]{x^2y}}$

d) $\dfrac{2}{\sqrt{5}-2}$

e) $\dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

f) Rationalize the numerator for $\dfrac{\sqrt{x}-2}{4-x}$.

Answers

IV. Rational Exponents

What is the meaning of $9^{1/2}$ or $4^{3/2}$ or $x^{3/4}$?

Definition: $a^{1/n} = \sqrt[n]{a}$; $\quad a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Any expression with a rational exponent can be rewritten in radical form. In certain situations this is quite helpful.

Illustration: Evaluate $9^{1/2}$, $4^{3/2}$, and $16^{-3/4}$.

Answers.

$$9^{1/2} = \sqrt{9} = 3$$

$$4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$$

$$16^{-5/4} = (\sqrt[4]{16})^{-5} = 2^{-5} = \frac{1}{32}$$

Although a calculator can do these problems, we suggest you only use one to check your results. Doing arithmetic (fractions, exponents, logs, etc.) by hand reinforces basic skills that you need for algebra.

Some problems that start in radical form are made easier by changing to rational exponents.

Examples:

a) $(\sqrt{3})^8 = (3^{1/2})^8 = 3^4 = 81$

b) $\sqrt[3]{2} \cdot \sqrt[4]{2} = 2^{1/3} \cdot 2^{1/4} = 2^{7/12}$

c) $\sqrt[3]{\dfrac{x^6}{y^{12}}} = \left(\dfrac{x^6}{y^{12}}\right)^{1/3} = \dfrac{x^2}{y^4}$

Exercise 4. Find the value of each (please no calculators).

a) 9^{-2} (b) $9^{1/2}$ (c) $4^{-3/2}$

d) $\left(\dfrac{8}{27}\right)^{-2/3}$ (e) $\left(\dfrac{81}{16}\right)^{1/4}$ Answers

Change to rational exponents, then simplify.

f) $(\sqrt[3]{2})^6$ (g) $\sqrt[3]{5} \cdot \sqrt[4]{5}$ (h) $\sqrt[6]{9}$ Answers

V. Logarithms

If you need an introduction or review of logorithms, see Review Topic 2, Section IV.

Evaluate: a) $\log_8 2$ b) $\log_2 \frac{1}{8}$

Answers:

a) $\log_8 2 = \frac{1}{3}$

$\log_8 2 = (\ \ ) \Rightarrow 8^{(\ \ )} = 2$. Since $\sqrt[3]{8} = 8^{1/3} = 2$, $(\ \ ) = \frac{1}{3}$.

b) $\log_2 \frac{1}{8} = -3$

$\log_2 \frac{1}{8} = (\ \ ) \Rightarrow 2^{(\ \ )} = \frac{1}{8}$. Since $2^{-3} = \frac{1}{8}$, $(\ \ ) = -3$.

Exercise 5. Evaluate the following.

a) $\log_3 \frac{1}{9}$

b) $\log_9 3$

c) $\log_8 2$

d) $\log_{10} 100$

e) $\log_{100} 10$

f) $\log_{100} \frac{1}{10}$

Answers

Express in simplest form.

a) $\sqrt{40}$

b) $-\sqrt{40}$

c) $\sqrt{-40}$

d) $\sqrt[3]{-40}$

e) $3\sqrt{5} - \sqrt{5}$

f) $(3\sqrt{5})^2$

g) $(\sqrt{3} - 2)(\sqrt{3} + 2)$

h) $\sqrt{\dfrac{20}{9}}$

Answers:

a) $2\sqrt{10}$

b) $-2\sqrt{10}$

c) Not a real number.

d) $-2\sqrt[3]{5}$

e) $2\sqrt{5}$

f) $9(5) = 45$

g) $(\sqrt{3})^2 - 2^2 = -1$

h) $\dfrac{\sqrt{20}}{3} = \dfrac{2\sqrt{5}}{3}$

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Express in simplest form.

a) $\sqrt{x^{16}}$ b) $\sqrt[3]{x^9}$ (c) $\sqrt{x^9}$

d) $\sqrt{54x^2y^3}$ e) $\sqrt[3]{54x^2y^3}$ (f) $(\sqrt{x}-\sqrt{y})^2$

g) $\sqrt{\left(\frac{x}{2}\right)^2-\left(\frac{\sqrt{3}x}{4}\right)^2}$ h) $\sqrt[3]{\frac{3x^9y^{10}}{24y^{-2}}}$ (i) $\sqrt{4x^2-20x+25}$

j) $\sqrt{4x^2-25}$ (k) True or False: is $\sqrt{x^2}=x$ for all values of x?

Answers:

a) $\sqrt{(x^8)^2}=x^8$ or $\sqrt{x^{16}}=(x^{16})^{1/2}=x^8$

b) $\sqrt[3]{(x^3)^3}=x^3$ or $\sqrt[3]{x^9}=(x^9)^{1/3}=x^3$

c) $\sqrt{x^9}=\sqrt{x^8\cdot x}=x^4\sqrt{x}$

(d) $\sqrt{(9x^2y^2)(6y)}=3xy\sqrt{6y}$

e) $\sqrt[3]{(27y^3)(2x^2)}=3y\sqrt[3]{2x^2}$

(f) $x-2\sqrt{xy}+y$

g) $\sqrt{\frac{x^2}{4}-\frac{3x^2}{16}}=\sqrt{\frac{x^2}{16}}=\frac{x}{4}$

(h) $\sqrt[3]{\frac{x^9y^{12}}{8}}=\frac{x^{9/3}y^{12/3}}{8^{1/3}}=\frac{x^3y^4}{2}$

i) $\sqrt{(2x-5)^2}=2x-5$

(j) Can't be simplified;
No repeated factors.

k) False. This statement is true as long as x is non-negative; $\sqrt{5^2}=5$ or $\sqrt{100^2}=100$, and so on. But what if $x=-5$?

$$\sqrt{x^2}=\sqrt{(-5)^2}=\sqrt{25}=5=-x,\ \text{not } x.$$

This leads to a more precise definition: $\sqrt{x^2}=|x|$. This is true for all x. In calculus you will be expected to use this definition.

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Rationalize. Consider all variables positive.

a) $\dfrac{4}{\sqrt{6}}$

b) $\sqrt{\dfrac{3}{2x}}$

c) $\dfrac{1}{\sqrt[4]{x^2y}}$

d) $\dfrac{2}{\sqrt{5}-2}$

e) $\dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

f) Rationalize the numerator for $\dfrac{\sqrt{x}-2}{4-x}$.

Answers:

a) $\dfrac{4}{\sqrt{6}} \cdot \dfrac{\sqrt{6}}{\sqrt{6}} = \dfrac{4\sqrt{6}}{6} = \dfrac{2}{3}\sqrt{6}$

b) $\sqrt{\dfrac{3}{2x}} = \dfrac{\sqrt{3}}{\sqrt{2x}} \cdot \dfrac{\sqrt{2x}}{\sqrt{2x}} = \dfrac{\sqrt{6x}}{2x}$

c) $\dfrac{1}{\sqrt[4]{x^2y}} \cdot \dfrac{\sqrt[4]{x^2y^3}}{\sqrt[4]{x^2y^3}} = \dfrac{\sqrt[4]{x^2y^3}}{xy}$

d) $\dfrac{2}{\sqrt{5}-2} \cdot \dfrac{\sqrt{5}+2}{\sqrt{5}+2} = \dfrac{2(\sqrt{5}+2)}{(\sqrt{5})^2-2^2} = 2(\sqrt{5}+2)$

e) $\dfrac{x-y}{\sqrt{x}-\sqrt{y}} \cdot \dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \dfrac{(x-y)(\sqrt{x}+\sqrt{y})}{x-y} = \sqrt{x}+\sqrt{y}$

f) $\dfrac{\sqrt{x}-2}{4-x} \cdot \dfrac{\sqrt{x}+2}{\sqrt{x}+2} = \dfrac{x-4}{(4-x)(\sqrt{x}+2)} = -\dfrac{1}{\sqrt{x}+2}$

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Find the value of each (please no calculators).

a) 9^{-2}

b) $9^{1/2}$

c) $4^{-3/2}$

d) $\left(\frac{8}{27}\right)^{-2/3}$

e) $\left(\frac{81}{16}\right)^{1/4}$

Change to rational exponents, then simplify.

f) $(\sqrt[3]{2})^6$

g) $\sqrt[3]{5} \cdot \sqrt[4]{5}$

h) $\sqrt[3]{\sqrt{2}}$

i) $\sqrt[6]{9}$

Answers:

a) $\frac{1}{81}$

b) 3

c) $2^{-3} = \frac{1}{8}$

d) $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$

e) $\frac{3}{2}$

f) $(2^{1/3})^6 = 2^2 = 4$

g) $5^{1/3} \cdot 5^{1/4} = 5^{7/12}$

h) $(2^{1/2})^{1/3} = 2^{1/6}$

i) $(3^2)^{1/6} = 3^{2/6} = 3^{1/3}$ or $\sqrt[3]{3}$

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Evaluate the following.

a) $\log_3 \dfrac{1}{9}$

b) $\log_9 3$

c) $\log_8 2$

d) $\log_{10} 100$

e) $\log_{100} 10$

f) $\log_{100} \dfrac{1}{10}$

Answers:

a) -2; $3^{-2} = \dfrac{1}{9}$

b) $\dfrac{1}{2}$; $9^{1/2} = 3$

c) $\dfrac{1}{3}$; $8^{1/3} = 2$

d) 2; $10^2 = 100$

e) $\dfrac{1}{2}$; $100^{1/2} = 10$

f) $-\dfrac{1}{2}$; $100^{-1/2} = 10^{-1} = \dfrac{1}{10}$

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