Math 109 – Topic 5

UNIT CIRCLE DEFINITION OF THE
TRIGONOMETRIC FUNCTIONS

In this section we derive the trigonometric functions based on the concept of a unit circle.

Consider a unit circle (radius $= 1$) centered at the origin. Topic 4 studied the formula $s = r\theta$. This relates the length of an arc of a circle with the radius of the circle and the central angle θ (in radians). For the unit circle, $r = 1$ and so $s = \theta$. That is, the arc length subtended ("marked off") by an angle is equal to the angle (in radians). We have the following picture, Fig. 5.1.
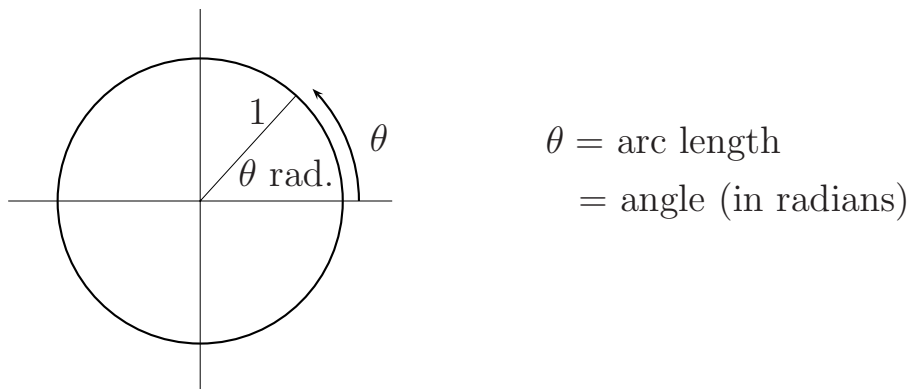

$$\theta = \text{arc length}$$
$$= \text{angle (in radians)}$$

Fig. 5.1. Unit Circle

Let (x, y) be the coordinates of a point P on the unit circle as it moves in a counterclockwise direction from the positive x-axis. The point P also determines an angle θ (in radians) which, as mentioned above, is equal to

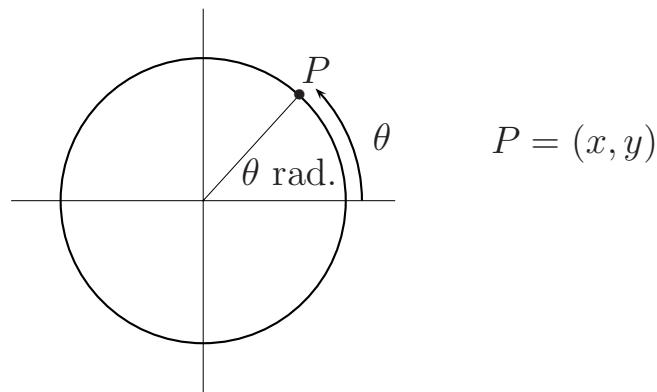the length of the arc traveled by P. See Fig. 5.2.

$$P = (x, y)$$

Fig. 5.2

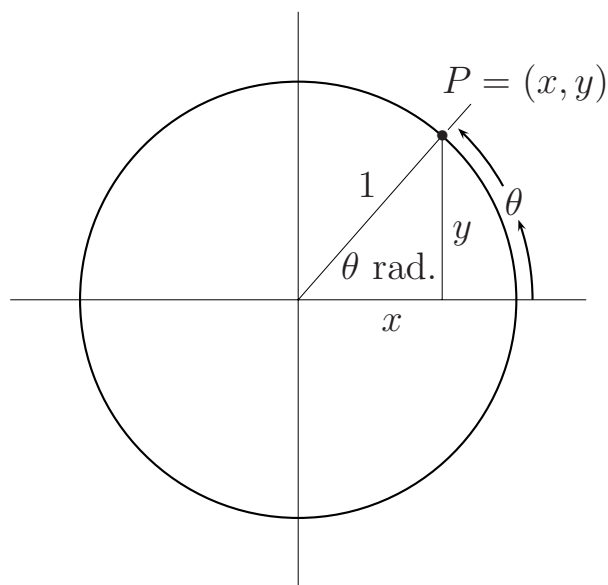Now, in this same circle, let us draw a right triangle as in Fig. 5.3.

Fig. 5.3

Using the right triangle definitions of the trigonometric functions (Topic 3a), we can write

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = y,$$

and

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = x.$$

In other words, we see that $\cos\theta = x$ and $\sin\theta = y$. Once $\sin\theta$ and $\cos\theta$ are determined, the other trigonometric functions are defined in the same manner as in the right triangle approach. This leads to an alternative definition of the trigonometric functions using the unit circle.

For a unit circle (radius $= 1$), a central angle measuring θ radians subtends an arc of length θ. This arc corresponds to a point P with coordinates (x, y). Then,

$$\cos\theta = x \quad \text{and} \quad \sin\theta = y.$$

That is, finding the cosine or sine of θ is the same as finding the x or y coordinate of P.

Maybe an example will help.

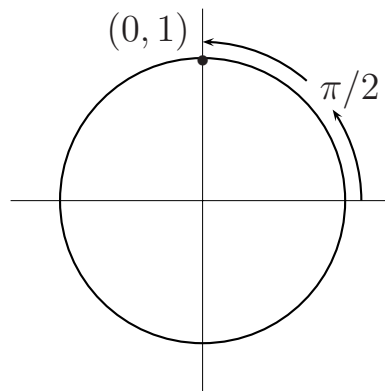**Example 5.1:** Find $\cos\dfrac{\pi}{2}$, $\sin\dfrac{\pi}{2}$ and $\tan\dfrac{\pi}{2}$.

Solution: $\theta = \dfrac{\pi}{2}$ corresponds to $P = (0, 1)$. Thus,

$$\cos\frac{\pi}{2} = x = 0,$$
$$\sin\frac{\pi}{2} = y = 1, \text{ and}$$
$$\tan\frac{\pi}{2} = \frac{y}{x} = \text{undefined},$$

since we cannot divide by 0.

In a similar fashion, $\theta = \pi$ corresponds to $P(-1, 0)$. This means $\cos\pi = x$ coordinate $= -1$ and $\sin\pi = y = 0$.

Exercise 1: Complete the table below.

θ	0	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$
$\sin\theta$				
$\cos\theta$				
$\tan\theta$				

Answer

There are other values of θ for which we know the x and y coordinates. Using the special angles discussed in Topic 3a, we can list the coordinates of P for $\theta = \dfrac{\pi}{6}, \dfrac{\pi}{4}$ and $\dfrac{\pi}{3}$ as indicated below.
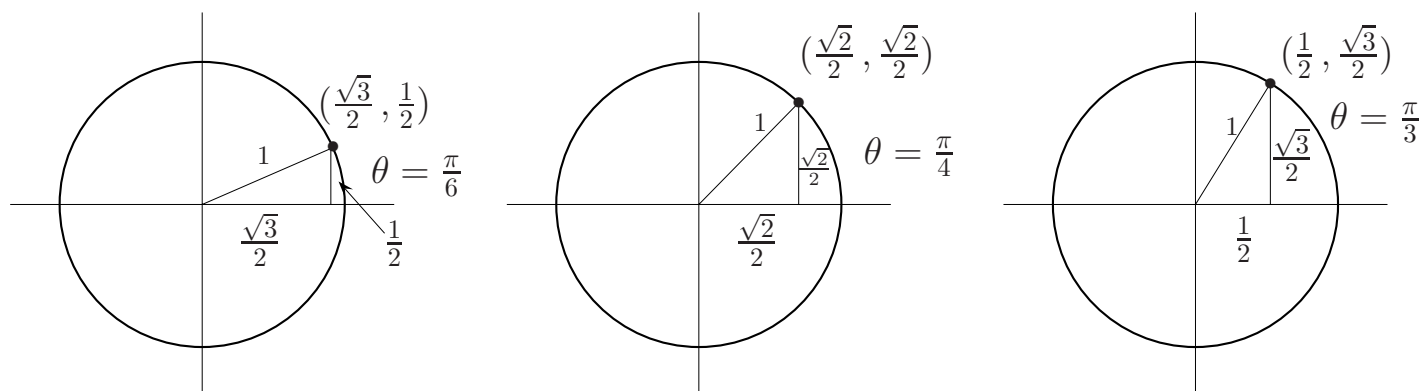

Fig. 5.4

This means $\cos\dfrac{\pi}{6} = x = \dfrac{\sqrt{3}}{2}$, $\sin\dfrac{\pi}{4} = y = \dfrac{\sqrt{2}}{2}$, etc, which matches the values obtained in the right angle definition.

As P moves from one quadrant to the next around the unit circle, the x and y coordinates will repeat themselves or have the opposite values of x and y coordinates in previous quadrants. For example, $\dfrac{\pi}{6}$ in the first quadrant matches the point $P\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$. Evidently, $\dfrac{5\pi}{6}, \dfrac{7\pi}{6}$, and $\dfrac{11\pi}{6}$ all correspond

to points whose coordinates are related to $P\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$ as indicated on the unit circle below.
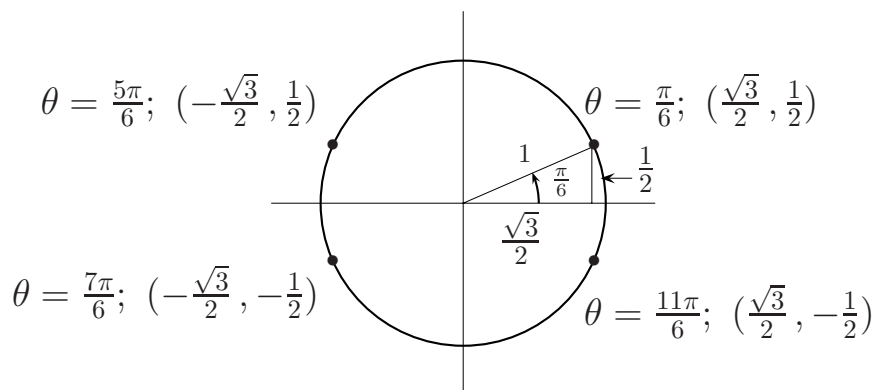

$$\theta = \tfrac{5\pi}{6};\ \left(-\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right) \qquad \theta = \tfrac{\pi}{6};\ \left(\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}\right)$$

$$\theta = \tfrac{7\pi}{6};\ \left(-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right) \qquad \theta = \tfrac{11\pi}{6};\ \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right)$$

Fig. 5.5

This means $\cos\dfrac{5\pi}{6} = x$ coordinate $= -\dfrac{\sqrt{3}}{2}$, $\sin\dfrac{5\pi}{6} = y$ coordinate $= \dfrac{1}{2}$, etc. Again, this agrees with what we learned in our discussion of reference angles and quadrant sign changes in Review Topic 9d. Similar comments hold for $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ and for $\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$.

The argument is exactly the same if θ is a negative angle. For example, if $\theta = -\dfrac{\pi}{6}$, then θ has the same x and y coordinates as $\theta = \dfrac{11\pi}{6}$ in Fig. 5.5. This means $\cos\left(-\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$ and $\sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$. In a similar fashion, if $\theta = -\dfrac{5\pi}{6}$, which gives the same point P as $\theta = \dfrac{7\pi}{6}$ in Fig. 5.5, then $\cos\left(-\dfrac{5\pi}{6}\right) = \dfrac{-\sqrt{3}}{2}$ and $\sin\left(\dfrac{-5\pi}{6}\right) = -\dfrac{1}{2}$.

The unit circle definition also enables us to determine when the trigonometric functions assume positive or negative values. For example, since the y coordinate > 0 in the first and second quadrants, we conclude that $\sin\theta$ (the y coordinate)> 0 if $0 < \theta < \pi$. If $\pi < \theta < 2\pi$, then $\sin\theta < 0$ since the y coordinate is negative for these values. Similarly, $\cos\theta$ (the x coordinate) is positive if $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ and negative if $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$. This gives us another way to compute exact values.

Periodicity is a property of the trigonometric functions that follows easily from the unit circle definition. The x and y coordinates of a point on the unit circle repeat themselves after one complete revolution or 2π radians. This means that $\cos\theta$ has the same values for θ and $\theta \pm 2\pi, \theta \pm 4\pi$, etc. A similar statement holds for $\sin\theta$. We say that $\cos\theta$ and $\sin\theta$ are periodic with period 2π. Thus,

$$\cos\left(\frac{13\pi}{3}\right) = \cos\left(\frac{\pi}{3} + 4\pi\right) = \cos\frac{\pi}{3} = \frac{1}{2}, \text{ and}$$

$$\cos\left(\frac{17\pi}{3}\right) = \cos\left(\frac{5\pi}{3} + 4\pi\right) = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}.$$

In each case, we rewrote the argument in terms of multiples of 2π and utilized the peridicity property.

The situation regarding periodicity is different for $\tan\theta$. Careful examination shows that the values of $\tan\theta$ start to repeat themselves after θ has traveled π radians. That is, the values of $\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ are the same as those for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. In other words, $\tan\theta$ is periodic with period π.

PRACTICE PROBLEMS for Topic 5 – (Unit Circle Definition of the Trigonometric Functions)

5.1 Indicate the quadrant(s) where θ terminates

a) $\sin\theta < 0$ c) $\tan\theta > 0$ and $\cos\theta < 0$

b) $\cos\theta > 0$ d) $\sec\theta = -\frac{4}{3}$ Answers

5.2 Without using a claculator, find exact values for the following:

a) $\sin 2\pi$,   b) $\sin 5\pi$,   c) $\tan\left(\frac{11\pi}{4}\right)$,   d) $\cos\left(\frac{17\pi}{6}\right)$,

e) $\sin\left(-\frac{11\pi}{6}\right)$.   Answers

5.3 Since exact values are so important, we've decided to give you additional practice. Print off as many copies as you need.

Group I

1) $\sin \pi$

2) $\sin \frac{3\pi}{2}$

3) $\cos \frac{\pi}{2}$

4) $\cos 2\pi$

5) $\tan \frac{\pi}{2}$

6) $\tan \frac{3\pi}{2}$

7) $\cos 3\pi$

8) $\sin \frac{5\pi}{2}$

9) $\cos \left(-\frac{\pi}{2}\right)$

10) $\sin(-\pi)$ $\hspace{1cm}$ Answers

Group II

1) $\sin \frac{\pi}{6}$

2) $\sin \frac{5\pi}{6}$

3) $\sin 330°$

4) $\cos \frac{\pi}{3}$

5) $\cos \frac{2\pi}{3}$

6) $\cos 300°$

7) $\tan \frac{\pi}{4}$

8) $\tan \frac{5\pi}{4}$

9) $\tan \frac{7\pi}{4}$

10) $\tan \frac{9\pi}{4}$ $\hspace{1cm}$ Answers

Group III

1) $\sin \frac{11\pi}{4}$

2) $\cos \frac{8\pi}{3}$

3) $\cos \left(-\frac{2\pi}{3}\right)$

4) $\tan \left(-\frac{3\pi}{4}\right)$

5) $\sin(-210°)$

6) $\tan \left(\frac{15\pi}{4}\right)$

7) $\sin \frac{23\pi}{4}$

8) $\cos \left(-\frac{17\pi}{6}\right)$

$\hspace{3cm}$ Answers

Group IV

1) $\frac{1}{\sin \frac{5\pi}{6}}$

2) $\csc \frac{5\pi}{6}$

3) $\frac{1}{\cos \frac{2\pi}{3}}$

4) $\sec \frac{2\pi}{3}$

5) $\frac{1}{\tan \frac{7\pi}{4}}$

6) $\cot \frac{7\pi}{4}$

7) $\csc \frac{3\pi}{2}$

8) $\sec 2\pi$ $\hspace{1cm}$ Answers

Beginning of Topic $\hspace{1cm}$ 109 Study Topics $\hspace{1cm}$ 109 Skills Assessment

Exercise 1

θ	0	$\dfrac{\pi}{2}$	π	$\dfrac{3\pi}{2}$
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	und	0	und

"und" means undefined.

Return to Review Topic

ANSWERS to SAMPLE PROBLEMS (Topic 5–Unit Circle Definition of the Trigonometric Functions)

5.1   a) III, IV   c) III

       b) I, IV   d) II, III

5.2.   a) 0;   b) 0;

       c) $\tan\left(\dfrac{11\pi}{4}\right) = \tan\left(\dfrac{3\pi}{4} + 2\pi\right) = \tan\left(\dfrac{3\pi}{4}\right) = -1;$

       d) $\cos\left(\dfrac{17\pi}{6}\right) = \cos\left(\dfrac{5\pi}{6} + 2\pi\right) = \cos\left(\dfrac{5\pi}{6}\right) = \dfrac{1}{2};$

       e) $\sin\left(\dfrac{-11\pi}{6}\right) = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}.$

5.3

Group I

1) $\sin\pi = 0$
2) $\sin\frac{3\pi}{2} = -1$
3) $\cos\frac{\pi}{2} = 0$
4) $\cos 2\pi = 1$
5) $\tan\frac{\pi}{2} =$ undef.
6) $\tan\frac{3\pi}{2} =$ undef.
7) $\cos 3\pi = \cos\pi = -1$
8) $\sin\frac{5\pi}{2} = \sin\frac{\pi}{2} = 1$
9) $\cos\left(-\frac{\pi}{2}\right) = 0$
10) $\sin(-\pi) = 0$

Group II

1) $\sin\frac{\pi}{6} = \frac{1}{2}$
2) $\sin\frac{5\pi}{6} = \frac{1}{2}$
3) $\sin 330° = -\frac{1}{2}$
4) $\cos\frac{\pi}{3} = \frac{1}{2}$
5) $\cos\frac{2\pi}{3} = -\frac{1}{2}$
6) $\cos 300° = \frac{1}{2}$
7) $\tan\frac{\pi}{4} = 1$
8) $\tan\frac{5\pi}{4} = 1$
9) $\tan\frac{7\pi}{4} = -1$
10) $\tan\frac{9\pi}{4} = \tan\frac{\pi}{4} = 1$

5.3   Continued

Group III

1) $\sin \frac{11\pi}{4} = \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

2) $\cos \frac{8\pi}{3} = \cos \frac{2\pi}{3} = -\frac{1}{2}$

3) $\cos \left(-\frac{2\pi}{3}\right) = \cos \frac{2\pi}{3} = -\frac{1}{2}$

4) $\tan \left(-\frac{3\pi}{4}\right) = -\tan \frac{3\pi}{4} = 1$

5) $\sin(-210°) = \sin 30° = \frac{1}{2}$

6) $\tan \left(\frac{15\pi}{4}\right) = \tan \frac{7\pi}{4} = -1$

7) $\sin \frac{23\pi}{4} = \sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$

8) $\cos \left(-\frac{17\pi}{6}\right) = \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

Group IV

1) $\frac{1}{\sin \frac{5\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$

2) $\csc \frac{5\pi}{6} = 2$

3) $\frac{1}{\cos \frac{2\pi}{3}} = \frac{1}{-\frac{1}{2}} = -2$

4) $\sec \frac{2\pi}{3} = -2$

5) $\frac{1}{\tan \frac{7\pi}{4}} = \frac{1}{-1} = -1$

6) $\cot \frac{7\pi}{4} = -1$

7) $\csc \frac{3\pi}{2} = \frac{1}{\sin^3 \frac{\pi}{2}} = -1$

8) $\sec 2\pi = \frac{1}{\cos 2\pi} = 1$