MATH 109 – TOPIC 7

GRAPHS OF TRIGONOMETRIC FUNCTIONS

I. Graphs of $\sin x$, $\cos x$, and $\tan x$

In Topic 3a, trigonometric "functions" were first introduced as ratios. In Topic 5, these same values were derived by considering the coordinates of points on the unit circle. This section will focus on trig functions and their graphs using a traditional algebraic approach.

The graphs of $\sin x$, $\cos x$, and $\tan x$ are given below, where x must be in radians. The table in Topic 6 is a summary of exact values that can be used to obtain these graphs.

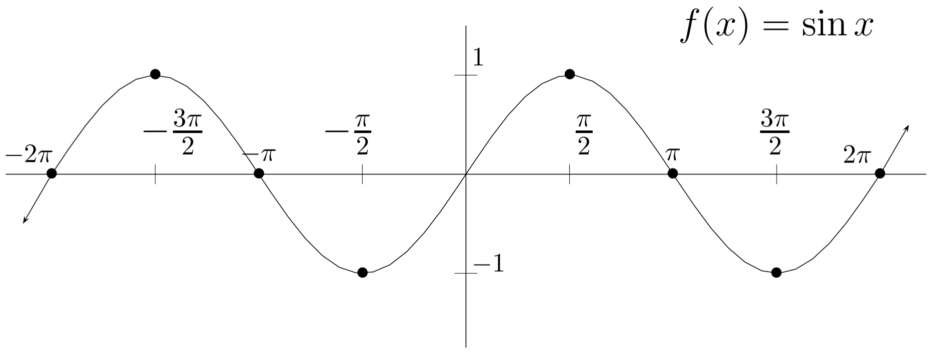


Fig. 7.1

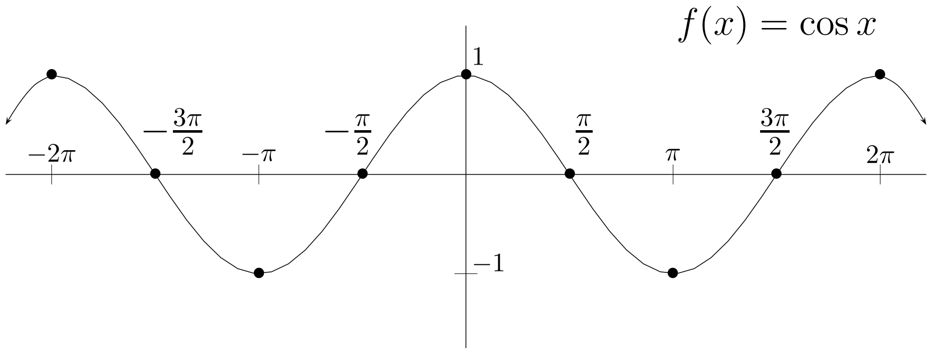


Fig. 7.2

Each function has a domain $(-\infty, \infty)$ and a range $[-1, 1]$. Both exhibit a repeatable pattern that is common to all trig functions. Such behavior is referred to as **periodic**. The period for $\sin x$ and $\cos x$ is 2π.

The graph of the tangent function combines the periodic behavior of a trig function (with period π) along with the algebraic behavior of a rational function. Using $\tan x = \dfrac{\sin x}{\cos x}$ (a fundamental identity covered in Topic 8), we can determine x intercepts from $\sin x = 0$ and vertical asymptotes from $\cos x = 0$. The resulting graph appears below.

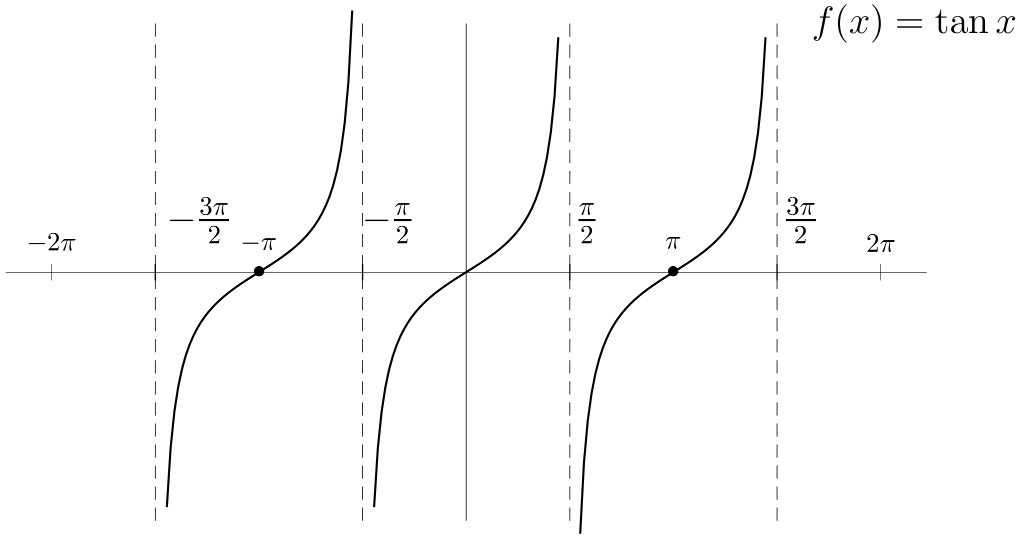


Fig. 7.3

Special notation is used to describe behavior near asymptotes:

as $x \to \dfrac{\pi}{2}^+$, $f(x) \to \infty$ (as x approaches $\frac{\pi}{2}$ from the right, values of the functions approach infinity)

as $x \to \dfrac{\pi}{2}^-$, $f(x) \to -\infty$

The concept of even and odd functions also applies here. Evidently, $\sin x$ and $\tan x$ are odd functions since they are symmetric with respect to the origin. Thus

$$\sin(-x) = -\sin x, \text{ and } \tan(-x) = -\tan x.$$

Translation: Opposite inputs yield opposite outputs.

Example 7.1. $\sin\left(-\dfrac{5\pi}{6}\right) = -\sin\left(\dfrac{5\pi}{6}\right) = -\dfrac{1}{2}$, and $\tan\left(-\dfrac{3\pi}{4}\right) = -\tan\left(\dfrac{3\pi}{4}\right) = -(-1) = 1$. (Review Topic 6)

The graph of $\cos x$ implies it is an even function since it is symmetric with respect to the y axis. This means

$$\cos(-x) = \cos x.$$

Example 7.2. $\cos\left(-\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$

The table below summarizes the previous discussion.

	Period	Symmetry	Domain	Range
$\sin x$	2π	odd	$(-\infty, \infty)$	$[-1, 1]$
$\cos x$	2π	even	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	π	odd	$x \neq$ odd multiple of $\pi/2$	$(-\infty, \infty)$

Table 7.4

PRACTICE PROBLEM for Topic 7 – Graphs of $\sin x$, $\cos x$, and $\tan x$

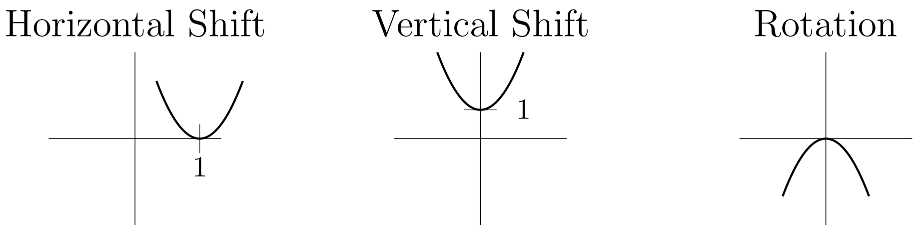
7.1 On a blank sheet of paper, by memory, draw the graphs of $\sin x$, $\cos x$, and $\tan x$ on the interval $-2\pi \leq x \leq 2\pi$. Label all x and y intercepts, maxima and minima (high and low points), and vertical asymptotes.

Answer. See Figures 7.1, 7.2, and 7.3.

II. Graphing by Transformations

Having just covered graphs of the basic trig functions, we will now examine graphs like $y = -\sin x$, $y = 1 + \cos x$ and $y = \tan 2x$. Functions such as these provide the opportunity to make use of transformations.

What do we mean by a transformation? An illustration may help. Here are 3 transformations of the graph $y = x^2$.



Exercise 1. Try writing an equation for each graph above. Answers

Transformations fall into one of two categories:

Output Based: includes vertical shifts, stretches, and rotations

Input Based: includes horizontal shifts, expansions, and compressions

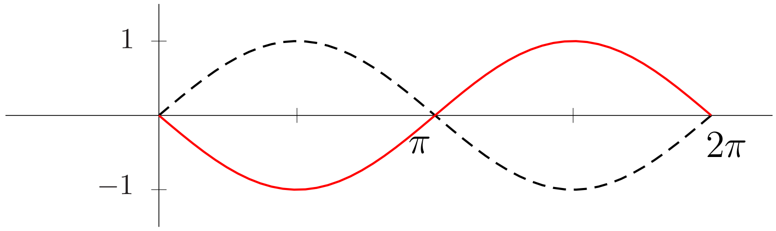
A. Output Based Transformations

Example 7.3. Sketch graphs of $y = -\sin x$, $y = 2\sin x$, and $y = 2 + \sin x$

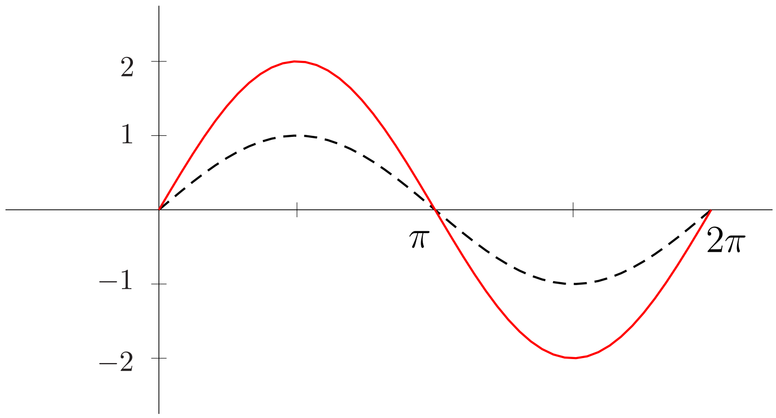
Below is a table of outputs for all 3 functions as well as $y = \sin x$.

x	$\sin x$	$-\sin x$	$2\sin x$	$2 + \sin x$
0	0	0	0	2
$\frac{\pi}{2}$	1	-1	2	3
π	0	0	0	2
$\frac{3\pi}{2}$	-1	1	-2	1
2π	0	0	0	2

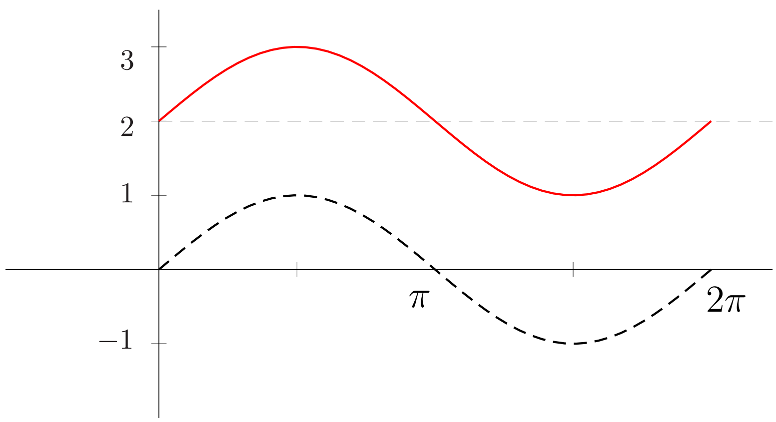
Before looking ahead, what do you expect from each transformation?



<u>$\sin x$ vs $-\sin x$</u>
rotation about the y axis.



<u>$\sin x$ vs $2 \sin x$</u>
vertical stretch by factor of 2.



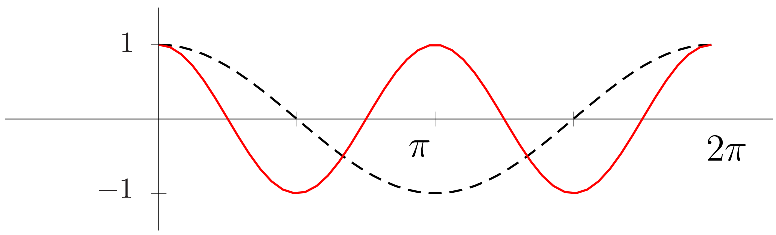
<u>$\sin x$ vs $2 + \sin x$</u>
vertical shift of $+2$ units.

Notice the period remains 2π radians. Output based transformations do not change the period.

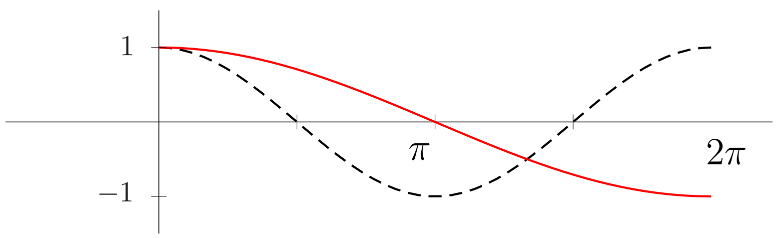
B. Input Based Transformations

Example 7.4. Sketch graphs of $y = \cos \dfrac{1}{2}x$, $y = \cos 2x$, and $y = \cos\left(x - \dfrac{x}{2}\right)$

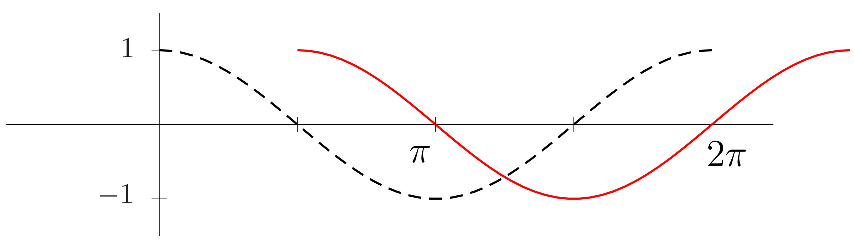
Tables are not as informative. Instead, let's examine each graph and write down our observations.



$\underline{\cos x \text{ vs } \cos 2x}$
Compression by factor of 2.
period: π



$\underline{\cos x \text{ vs } \cos \frac{1}{2}x}$
expansion by factor of 2
period: 4π



$\underline{\cos x \text{ vs } \cos(x - \frac{\pi}{2})}$
horizontal shift of $+\pi/2$ units
period: 2π

Notice that input based transformations have no effect on range.

PRACTICE PROBLEM for Topic 7 – Graphing by Transformations

7.2 Using transformations, sketch a graph of each:

a) $y = 1 + \cos x$

b) $y = -3 \sin x$

c) $y = \tan\left(x + \frac{\pi}{4}\right)$

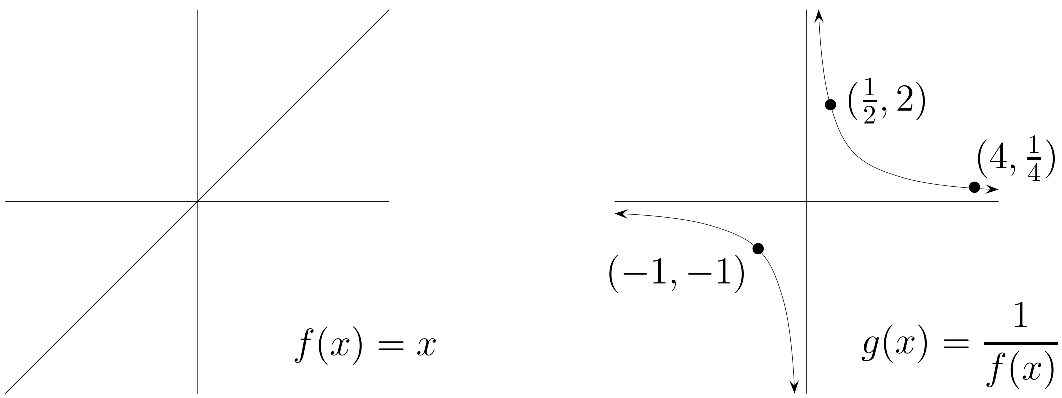
d) $y = \cos 2x$

e) $y = 2 - \sin x$
(Hint: Requires 2 transformations)

Answers

III. Reciprocals

What's the difference between a transformation and a reciprocal?
After a shift, stretch, or rotation, a sine function will still be recognizable as a sine function. A reciprocal, however, looks entirely different. To clarify this point, let's compare the graphs of $f(x) = x$ with its reciprocal.



Which of the following characterize a reciprocal graph?

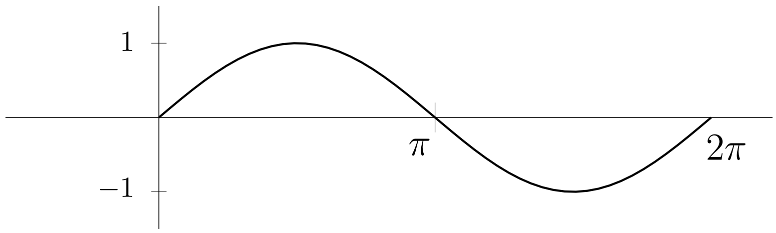
1) Over similar intervals, $f(x)$ and $\frac{1}{f(x)}$ have the same sign.

2) As $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$. Such behavior (also known as end behavior) indicates a horizontal asymptote.

3) If $f(c) = 0$, then $\frac{1}{f(c)}$ is undefined and the line $x = c$ is a vertical asymptote.

All are correct (and helpful when graphing a reciprocal).

Exercise 2. Sketch the graph of $y = \dfrac{1}{\sin x} = \csc x$.
Start with $y = \sin x$ and identify characteristics of its reciprocal.



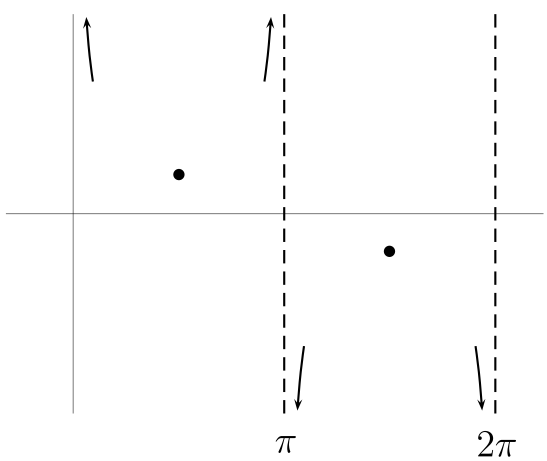
$y = \sin x$

1) On $(0, \pi)$ $\sin x$ and its reciprocal are positive.
   On $(\pi, 2\pi)$ both functions are negative.

2) For all x, $|\sin x| \le 1 \Rightarrow |\csc x| \ge 1$; $\csc \dfrac{\pi}{2} = 1$, $\csc \dfrac{3\pi}{2} = -1$.

3) Because $\sin 0 = \sin \pi = \sin 2\pi = 0$, $\csc x$ has vertical asymptotes at $x = 0,\ \pi,\ 2\pi$.

Try completing the reciprocal graph we've started.



Answer

PRACTICE PROBLEM for Topic 7 – Reciprocals

7.3 Using our hints for graphing reciprocals, try sketching $y = \dfrac{1}{\cos x} = \sec x$ and $y = \dfrac{1}{\tan x} = \cot x$.

Answer

Beginning of Topic 109 Study Topics 109 Skills Assessment

Exercise 1

Horizontal Shift
$y = (x-1)^2$;

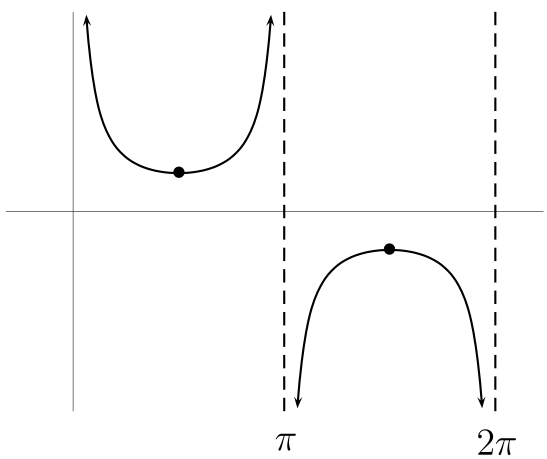
Vertical Shift
$y = x^2 + 1$;

Rotation
$y = -x^2$

Return to Review Topic

Exercise 2

Completing the graph of $y = \csc x$ (reciprocal of $y = \sin x$):



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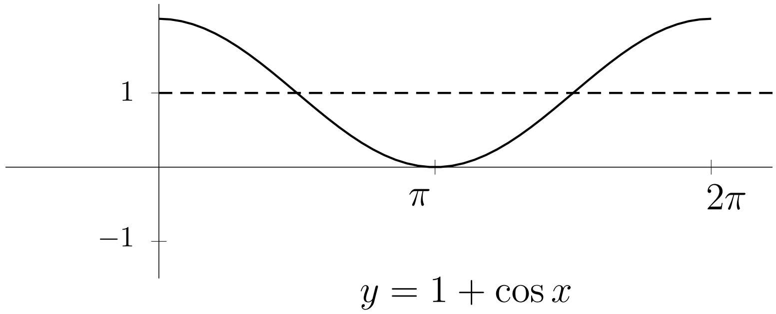
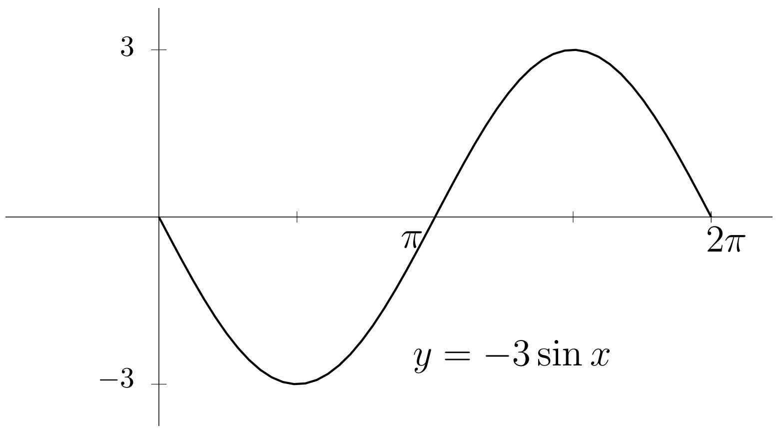
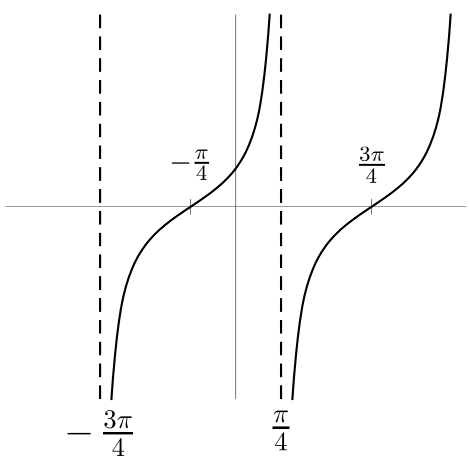
7.2 a)



b)



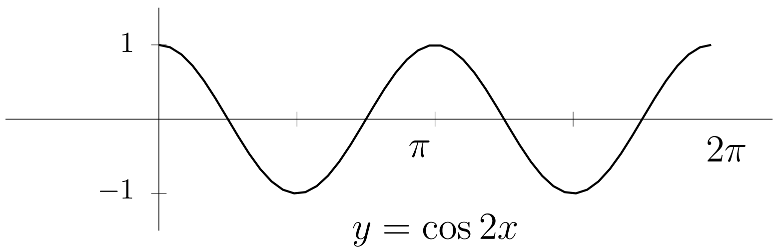
c)



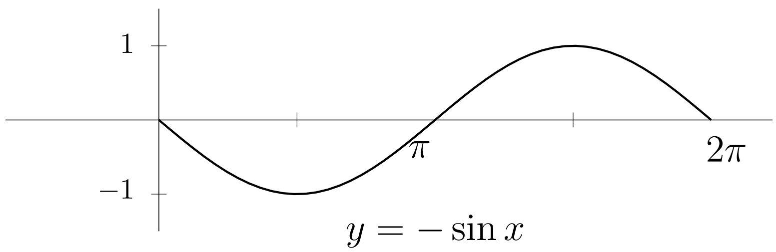
Return to Problem

7.2 Continued

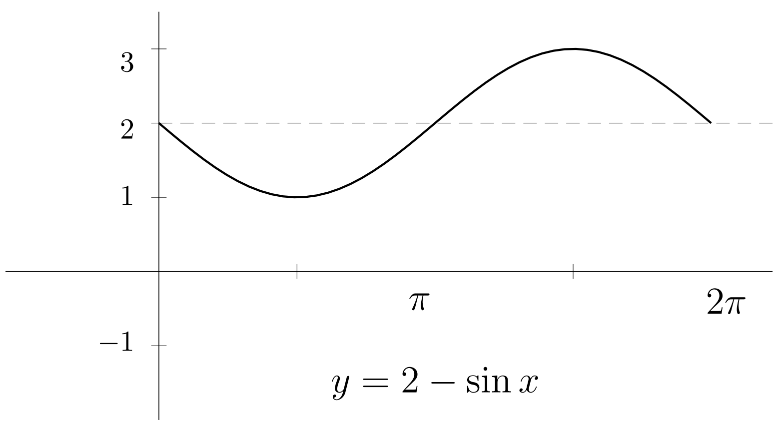
d)



e)



Rotate about y axis, then vertically shift.



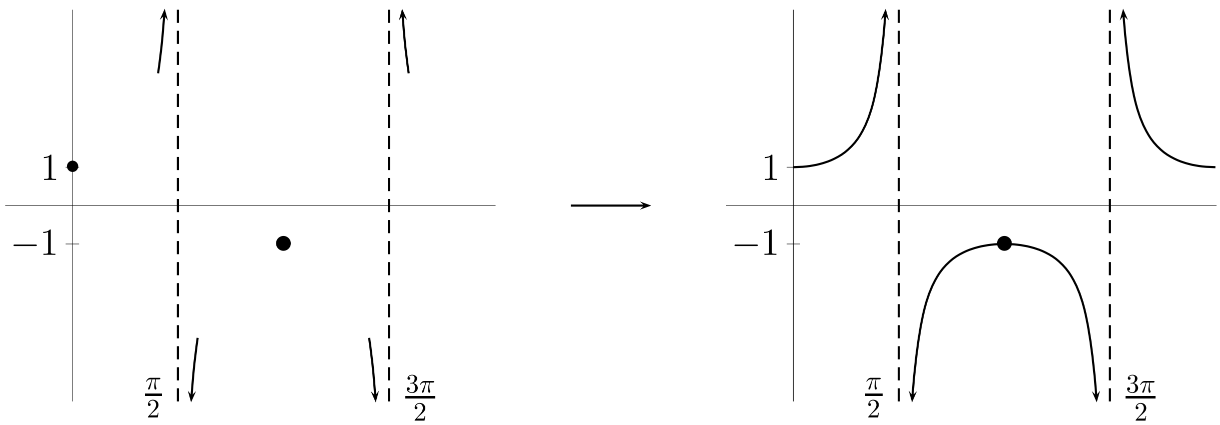
Return to Problem

7.3 a) Graph $y = \sec x$. From the graph of $\cos x$ we learn:

1) $\cos x$ and $\sec x > 0$ on $[0, \dfrac{\pi}{2})$ and $(\dfrac{3\pi}{2}, 2\pi]$;

2) $|\cos x| \leq 1 \Rightarrow |\sec x| \geq 1$, $\cos 0 = \sec 0 = 1$; $\cos \pi = \sec \pi = -1$;

3) Because $\cos x = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$, $\sec x$ has vertical asymptotes at $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$.



Part B continued on next page.

Return to Problem

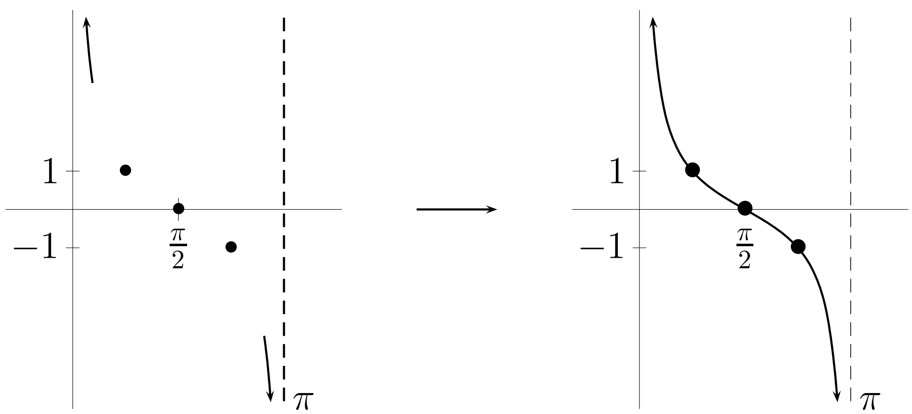
7.3 Continued

b) Graph $y = \cot x$. From the graph of $\tan x$ we learn:

1) $\tan x$ and $\cot x > 0$ on $(0, \frac{\pi}{2})$ and $(\pi, \frac{3\pi}{2})$
$\tan x$ and $\cot x < 0$ on $(\frac{\pi}{2}, \pi)$ and $(\frac{3\pi}{2}, 2\pi)$;

2) Range of $\tan x$ and $\cot x$: $(-\infty, \infty)$; $\tan \frac{\pi}{4} = \cot \frac{\pi}{4} = 1$;
$\tan \frac{3\pi}{4} = \cot \frac{3\pi}{4} = -1$

3) $\tan 0 = \tan \pi = 0 \Rightarrow \cot x$ has VA at $x = 0,\ \pi$.
$\tan \frac{\pi}{2}$ is undefined $\Rightarrow \cot \frac{\pi}{2} = 0$.



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