MATH 109 – TOPIC 8
TRIGONOMETRIC IDENTITIES

I. Using Algebra in Trigonometric Forms

Practice Problems

II. Verifying Identities

Practice Problems

I. Using Algebra in Trigonometric Form

This topic will focus mainly on identities and how they can be verified. That means you will have to memorize (or be able to derive) a list of the fundamental identities (see Part II of this topic). Just as important will be your ability to recognize and perform various algebraic processes (factoring, multiplying polynomials, simplifying rational expressions, . . .) with trig-type expressions. This is not new algebra, but it probably looks different.

Here are examples of some basic operations; both algebra and trig versions.

1. Combining Like Terms

$$x^2 + 3x^2 = 4x^2 \qquad\qquad \sin\theta + 3\sin\theta = 4\sin\theta$$

Common error: $\sin\theta + \sin 2\theta \neq \sin 3\theta$

In trig expressions, like terms must have the same angle (better known as argument).

2. Simplification of Fractions

a) $\dfrac{x^2}{3x} = \dfrac{x}{3}$ $\dfrac{\tan^2\theta}{3\tan\theta} = \dfrac{\tan\theta}{3}$

Common error: $\dfrac{\tan 2\theta}{\tan\theta} \neq 2$

b) $\dfrac{1-x^2}{1-x} = \dfrac{(1-x)(1+x)}{1-x}$ $\dfrac{1-\cos^2 x}{1-\cos x} = \dfrac{(1-\cos x)(1+\cos x)}{1-\cos x}$

$= 1+x$ $= 1+\cos x$

Simplifying a fraction containing a sum requires factoring. **Only factors can be cancelled.**

Common error: $\dfrac{\sin^2\theta+\cos\theta}{\sin\theta} \neq \sin\theta+\cos\theta$

3. Multiplication of Polynomials

a) $2x(x^2) = 2x^3$ $\sin x(\sin^2 x) = \sin^3 x$

Remember that exponents count factors: $\sin^3 x$ means that $\sin x$ is a factor 3 times.

Common error:

$$\sin x(\sin^2 x) \neq \sin^3 x^2$$
$$\sin x(\sin 2x) \neq \sin 2x^2$$

b) $(2x)^2 = 4x^2$ $(\sin 2x)^2 = \sin^2 2x$

c) $2x(x-3) = 2x^2 - 6x$ $\sin x(\tan x - \sec x)$

$= \sin x\tan x - \sin x\sec x$

The most common error: $\sin(a+b) \neq \sin a + \sin b$
sin ( ) represents a composite function where the inner function [whatever appears inside the ( )] is being placed inside a sin function.

What does $\sin(a+b)$ equal? Aren't you curious? If you can't wait to find out, go to the list of basic identities on pg. 6, Topic 8-II.

d) $(x-\sqrt{2})^2 = x^2 - 2\sqrt{2}x + 2$ $\quad (\sin 2x - \cos x)^2$

$= \sin^2 2x - 2\sin 2x \cos x + \cos^2 x$

Common error: $(\sin x - \cos x)^2 \neq \sin^2 x + \cos^2 x$. Squaring a binomial always yields 3 terms.

e) $(2x-3)(x+4)$ $\quad (\sin x - 2)(\sin x - 1)$

$= 2x^2 + 5x - 1$ $\quad = \sin^2 x - 3\sin x + 2$

4. <u>Factoring</u>

a) Greatest Common Factor

$x^2 - 3x = x(x-3)$ $\quad \sin^2 x - \sin x \cos x$

$= \sin x(\sin x - \cos x)$

Common error: $\sin a + \sin b \neq \sin(a+b)$. Sin ( ) is a function not a product of factors.

b) Difference of Squares and Trinomials

$1 - x^2 = (1-x)(1+x)$ $\quad \tan^2 x - 1 = (\tan x - 1)(\tan x + 1)$

$x^2 - 3x - 4$ $\quad \sin^2 x - 3\sin x - 4$

$= (x-4)(x+1)$ $\quad = (\sin x - 4)(\sin x + 1)$

Common error: $1 + \sin^2 x \neq (1 + \sin x)^2$

The sum of 2 squares is prime.

This is just a start. Recognizing algebra techniques when they occur in trig (or any other form) is going to take time, constant comparisons with more familiar forms, and lots of practice. Speaking of practice, here's a start. (Save the "thank-you's" for the next time we meet.)

Practice Problems

8.1. True or False

a) $\sin\theta + \sin 3\theta = \sin 4\theta$

b) $2\sin\theta \cdot \sin\theta = 2\sin\theta^2$

c) $\sin 2\theta = 2\sin\theta$

d) $\dfrac{\sin 3\theta}{\sin\theta} = \sin 3$

e) $\dfrac{\sin 3\theta}{\theta} = \sin 3$

f) $\dfrac{\sin^3 2\theta}{\sin 2\theta} = \sin^2 2\theta$

Answers

8.2. Express the following in factored form

a) $\sin x\cos x - \sin^2 x$

b) $1 - 2\cos x + \cos^2 x$

c) $\sin^2 x - 5\sin x + 4$

d) $1 - \sin^2 4x$

Answers

8.3 Expanding Binomials

a) $(\sin x + \cos x)^2$

b) $(\sin 2x - \cos 2x)^2$

c) $(1 + \tan^2 x)^2$

d) $(1 + \tan^2 x)^3$

e) $(1 + \tan x^2)^2$

Answers

II. Verifying Identities

This section discusses trigonometric identites. It may be helpful first to review what the word "identity" means.

For our purposes, an identity is an expression involving an equation and a variable. Most importantly, to be an identity the equation must be true for any defined value of the variable. For example, the following equation is not an identity because equality holds only when x=2.

$$2x - 1 = 3 \qquad \text{(Not an identity)}$$

One of the most well-known trigonometric identities is the following.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{(An identity)}$$

The above equation is true for any value of θ, and so it is an identity. This means that the expression $(\sin^2\theta + \cos^2\theta)$ can be replaced by 1 and alternatively, the number 1 can be replaced by the expression $(\sin^2\theta + \cos^2\theta)$. Let's actually derive this identity.

In Topic 5, we saw that $\cos\theta$ and $\sin\theta$ can be considered as the x and y coordinates of a point P on the unit circle. Fig. 8.1 below illustrates this.

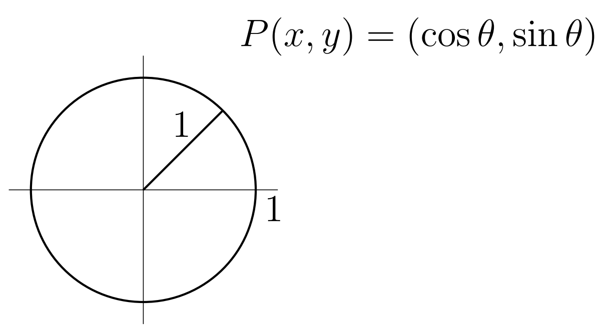

Fig. 8.1.

From the Pythagorean Theorem, we have that

$$x^2 + y^2 = 1, \text{ or} \tag{1}$$
$$\cos^2\theta + \sin^2\theta = 1. \tag{2}$$

Clearly, Eqn (1) is true for any point $P(x,y)$ on the unit circle. This means Eqn (2) holds for any value of θ and is thus an identity.

In case you're still not convinced, let's select 2 values of θ and show that $\sin^2\theta + \cos^2\theta = 1$.

a) Given $\theta = \frac{\pi}{6}$, $\sin\theta = \frac{1}{2}$, and $\cos\theta = \frac{\sqrt{3}}{2}$.

Then $\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.$

b) Given $\theta = \frac{3\pi}{4}$, $\sin\theta = \frac{1}{\sqrt{2}}$, and $\cos\theta = -\frac{1}{\sqrt{2}}$.

Then $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1.$

Listed below are some (but not all!) of the basic identities that may be used in first semester calculus. Their derivations can be found in any trigonometry book. Note that a complete listing of trigonometric identities and other related information can be found in Topic 13.

Basic Forms	Alternate Forms
a) $\sin^2 a + \cos^2 a = 1;$	$\sin^2(\ ) + \cos^2(\ ) = 1$
b) $\tan^2 a + 1 = \sec^2 a;$	$\tan^2(\ ) + 1 = \sec^2(\ )$
c) $\cot^2 a + 1 = \csc^2 a;$	$\cot^2(\ ) + 1 = \csc^2(\ )$
d) $\sin(a \pm b) = \sin a\cos b \pm \cos a \sin b;$	$\sin((\ )_1 \pm (\ )_2) = \sin(\ )_1\cos(\ )_2 \pm \cos(\ )_1\sin(\ )_2$
e) $\sin(2a) = 2\sin a\cos a;$	$\sin(2(\ )) = 2\sin(\ )\cos(\ )$
f) $\cos(a \pm b) = \cos a\cos b \mp \sin a \sin b;$	$\cos((\ )_1 \pm (\ )_2) = \cos(\ )_1\cos(\ )_2 \mp \sin(\ )_1\sin(\ )_2$
g) $\cos(2a) = \cos^2 a - \sin^2 a;$ $=2\cos^2 a - 1;$ $=1 - 2\sin^2 a;$	$\cos(2(\ )) = \cos^2(\ ) - \sin^2(\ )$ $=2\cos^2(\ ) - 1$ $=1 - 2\sin^2(\ )$

h) $\cos^2 a = \dfrac{1+\cos(2a)}{2}$; $\qquad \cos^2(\ ) = \dfrac{1+\cos(2(\ ))}{2}$

i) $\sin^2 a = \dfrac{1-\cos(2a)}{2}$; $\qquad \sin^2(\ ) = \dfrac{1-\cos(2(\ ))}{2}$

j) $\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$; $\qquad \tan((\ )_1 \pm (\ )_2) = \dfrac{\tan(\ )_1 \pm \tan(\ )_2}{1 \mp \tan(\ )_1 \tan(\ )_2}$

k) $\tan(2a) = \dfrac{2\tan a}{1-\tan^2 a}$; $\qquad \tan(2(\ )) = \dfrac{2\tan(\ )}{1-\tan^2(\ )}$

In solving problems involving trigonometric identities, you must be able to recognize the identities when they appear. Therefore we recommend memorizing the basic forms listed above. Since memory can sometimes fail, it may be useful to understand how some of them are related. For example, dividing (a) by $\cos^2 a$ gives (b), while dividing (a) by $\sin^2 a$ implies (c).

The alternate forms of the above identities are really more useful than the basic forms. They allow us to apply the identities in many situations that might not otherwise be apparent.

For example, using the alternative form of identity (e) we can write:

$$\begin{aligned}
\sin 6\theta = \sin 2(3\theta) = \sin 2(\ ) &= 2\sin(\ )\cos(\ ) \\
&= 2\sin 3\theta \cos 3\theta, \text{ or} \\
\sin 6\theta &= 2\sin 3\theta\cos 3\theta
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sin 8\theta &= 2\sin 4\theta\cos 4\theta, \\
\sin 10\theta &= 2\sin 5\theta\cos 5\theta, \text{ etc.}
\end{aligned}$$

(Now you see why it's called the "double-angle" formula.)

As another example, consider $\sin 3\theta$. Using the alternate form of identity (d), we have:

$$\begin{aligned}
\sin 3\theta &= \sin((2\theta)+(\theta)) = \sin((\ )_1 + (\ )_2) \\
&= \sin(\ )_1\cos(\ )_2 + \cos(\ )_1\sin(\ )_2, \text{ or} \\
\sin 3\theta &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta
\end{aligned}$$

Remark: The alternate forms of the identities merely use blank parentheses. This idea of using blank parentheses instead of letters or symbols can be extremely beneficial regardless of the course or setting or problem. We strongly recommend that you try to use this idea in all your math classes. (If this remark does not make sense right now, ask your teacher about it when you have a chance. It's that important!)

The basic identities listed earlier along with the definitions of the trigonometric functions (Topic 3a) can be used to verify other identities. For example, verify that

$$\frac{\sin\theta}{1-\sin^2\theta} = \sec\theta\tan\theta.$$

One solution technique is to start with one side and work toward the other using known information. Here, we begin with the right side.

$$\sec\theta\tan\theta = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{1-\sin^2\theta}.$$

A slightly more difficult example is the following. Verify

$$\frac{1+\tan\theta}{1+\cot\theta} = \tan\theta.$$

Let's begin with the more complicated expression on the left.

$$\begin{aligned}
\frac{1+\tan\theta}{1+\cot\theta} &= \frac{1+\tan\theta}{1+\dfrac{1}{\tan\theta}} && \text{(definition of } \cot\theta\text{)}\\
&= \frac{1+\tan\theta}{\dfrac{\tan\theta+1}{\tan\theta}} && \text{(adding fractions in the denominator)}\\
&= (1+\tan\theta)\div\left(\frac{\tan\theta+1}{\tan\theta}\right) && \text{(equivalent form)}\\
&= (1+\tan\theta)\cdot\frac{\tan\theta}{(\tan\theta+1)} && \text{(definition of division)}\\
&= \tan\theta.
\end{aligned}$$

In working with identities, sometimes writing everything in terms of $\sin\theta$ or $\cos\theta$ may help. Let's try that previous example again using this approach.

$$\frac{1+\tan\theta}{1+\cot\theta} = \frac{1+\dfrac{\sin\theta}{\cos\theta}}{1+\dfrac{\cos\theta}{\sin\theta}} \qquad \text{(definitions)}$$

$$= \frac{\dfrac{\cos\theta+\sin\theta}{\cos\theta}}{\dfrac{\sin\theta+\cos\theta}{\sin\theta}} \qquad \text{(adding fractions)}$$

$$= \frac{\cos\theta+\sin\theta}{\cos\theta} \div \frac{\sin\theta+\cos\theta}{\sin\theta} \qquad \text{(equivalent form)}$$

$$= \frac{(\cos\theta+\sin\theta)}{\cos\theta} \cdot \frac{\sin\theta}{(\sin\theta+\cos\theta)} \qquad \text{(definition of division)}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

In calculus, you will not be asked to verify identities. Instead, you will start with an expression and, using identities, manipulate it into an equivalent form more useful for the problem at hand. Two situations where this need will arise involve solving trigonometric equations and performing a process called integration.

Practice Problems.

8.4 Verify the following identities.

a) $\sin\theta\sec\theta = \tan\theta$

b) $\sin^2\theta = \tan\theta\cot\theta - \cos^2\theta$ Answers

c) $\sin\theta(\cot\theta+\tan\theta) = \sec\theta$

d) $\sec^2\theta\csc^2\theta = \sec^2\theta+\csc^2\theta$ Answers

e) $\cos^2 2\theta - \sin^2 2\theta = \cos 4\theta$ Answer

8.1. a) False

b) False

c) False

d) False

e) False

f) True Return to Problem

8.2. a) $\sin x(\cos x - \sin x)$

b) $(1 - \cos x)^2$

c) $(\sin x - 1)(\sin x - 4)$

d) $(1 - \sin 4x)(1 + \sin 4x)$ Return to Problem

8.3. a) $\sin^2 x + 2\sin x \cos x + \cos^2 x$

b) $\sin^2 2x - 2\sin 2x \cos 2x + \cos^2 2x$

c) $1 + 2\tan^2 x + \tan^4 x$

d) $1 + 3\tan^2 x + 3\tan^4 x + \tan^6 x$

e) $1 + 2\tan x^2 + \tan^2 x^2$ Return to Problem

8.4. a) $\sin\theta \sec\theta = \sin\theta \cdot \dfrac{1}{\cos\theta} = \tan\theta$

b) $\tan\theta \cot\theta - \cos^2\theta = \tan\theta \cdot \dfrac{1}{\tan\theta} - \cos^2\theta = 1 - \cos^2\theta = \sin^2\theta$

Return to Problem

8.4. c)

$$\begin{aligned}
\sin\theta(\cot\theta+\tan\theta) &= \sin\theta\left(\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\right)\\
&= \sin\theta\left(\frac{\cos\theta\cos\theta}{\sin\theta\cos\theta}+\frac{\sin\theta\sin\theta}{\cos\theta\sin\theta}\right)\\
&= \sin\theta\left(\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cdot\cos\theta}\right)=\frac{\sin\theta(1)}{\sin\theta\cos\theta}\\
&= \frac{1}{\cos\theta}=\sec\theta
\end{aligned}$$

d)

$$\begin{aligned}
\sec^2\theta+\csc^2\theta &= \frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}\\
&= \frac{\sin^2\theta}{\cos^2\theta\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}\\
&= \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}=\frac{1}{\sin^2\theta\cos^2\theta}\\
&= \frac{1}{\sin^2\theta}\cdot\frac{1}{\cos^2\theta}=\csc^2\theta\sec^2\theta
\end{aligned}$$

Return to Problem

8.4. e) To solve this problem, we first notice that the arguments are not the same. Nearly always, this means the expressions must be manipulated until all the arguments are the same. Typically, we begin with the expressions involving the larger arguments. The idea of using blank parentheses is also crucial here. Based on the alternate form of the first identity in (e), we have

$$\begin{aligned} \cos 4\theta &= \cos 2(2\theta) \\ &= \cos 2(\ \ ) = \cos^2(\ \ ) - \sin^2(\ \ ) = \cos^2(2\theta) - \sin^2(2\theta). \end{aligned}$$

Return to Problem