MATH 150 – TOPIC 14

TRIGONOMETRIC IDENTITIES

This section discusses trigonometric identites. It may be helpful first to review what the word "identity" means.

For our purposes, an identity is an expression involving an equation and a variable. Most importantly, to be an identity the equation must be true for <u>any</u> defined value of the variable. For example, the following equation is <u>not</u> an identity because equality holds only when $x = 2$.

$$2x - 1 = 3 \qquad\qquad \text{(Not an identity)}$$

One of the most well-known trigonometric identities is the following.

$$\sin^2\theta + \cos^2\theta = 1 \qquad\qquad \text{(An identity)}$$

The above equation is true for any value of θ, and so it is an identity. This means that the expression $(\sin^2\theta + \cos^2\theta)$ can be replaced by 1 and alternatively, the number 1 can be replaced by the expression $(\sin^2\theta+\cos^2\theta)$. Let's actually derive this identity.

In Review Topic 11, we saw that $\cos\theta$ and $\sin\theta$ can be considered as the x and y coordinates of a point P on the unit circle. Fig. 14.1 below illustrates this.

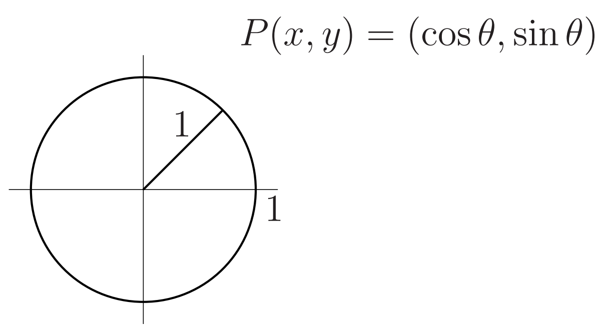

Fig. 14.1.

From the Pythagorean Theorem, we have that

$$x^2 + y^2 = 1, \text{ or} \tag{14.1}$$

$$\cos^2\theta + \sin^2\theta = 1. \tag{14.2}$$

Clearly, Eqn (14.1) is true for <u>any</u> point $P(x, y)$ on the unit circle. This means Eqn (14.2) holds for <u>any</u> value of θ and is thus an identity.

In case you're still not convinced, let's select 2 values of θ and show that $\sin^2\theta + \cos^2\theta = 1$.

a) Given $\theta = \dfrac{\pi}{6}$, $\sin\theta = \dfrac{1}{2}$, and $\cos\theta = \dfrac{\sqrt{3}}{2}$.

Then $\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{1}{4} + \dfrac{3}{4} = 1.$

b) Given $\theta = \dfrac{3\pi}{4}$, $\sin\theta = \dfrac{1}{\sqrt{2}}$, and $\cos\theta = -\dfrac{1}{\sqrt{2}}$.

Then $\left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(-\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2} + \dfrac{1}{2} = 1.$

Continued on next page.

Listed below are some (but not all!) of the basic identities that may be used in first semester calculus. Their derivations can be found in any trigonometry book. Note that a complete listing of trigonometric identities and other related information can be found in Review Topic 17.

Basic Forms	Alternate Forms
a) $\sin^2 a + \cos^2 a = 1;$	$\sin^2(\ ) + \cos^2(\ ) = 1$
b) $\tan^2 a + 1 = \sec^2 a;$	$\tan^2(\ ) + 1 = \sec^2(\ )$
c) $\cot^2 a + 1 = \csc^2 a;$	$\cot^2(\ ) + 1 = \csc^2(\ )$
d) $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b;$	$\sin((\ )_1 \pm (\ )_2) = \sin(\ )_1 \cos(\ )_2 \pm \cos(\ )_1 \sin(\ )_2$
e) $\sin(2a) = 2 \sin a \cos a;$	$\sin(2(\ )) = 2 \sin(\ ) \cos(\ )$
f) $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b;$	$\cos((\ )_1 \pm (\ )_2) = \cos(\ )_1 \cos(\ )_2 \mp \sin(\ )_1 \sin(\ )_2$
g) $\cos(2a) = \cos^2 a - \sin^2 a;$ $= 2\cos^2 a - 1;$ $= 1 - 2\sin^2 a;$	$\cos(2(\ )) = \cos^2(\ ) - \sin^2(\ )$ $= 2\cos^2(\ ) - 1$ $= 1 - 2\sin^2(\ )$
h) $\cos^2 a = \dfrac{1 + \cos(2a)}{2};$	$\cos^2(\ ) = \dfrac{1 + \cos(2(\ ))}{2}$
i) $\sin^2 a = \dfrac{1 - \cos(2a)}{2};$	$\sin^2(\ ) = \dfrac{1 - \cos(2(\ ))}{2}$
j) $\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b};$	$\tan((\ )_1 \pm (\ )_2) = \dfrac{\tan(\ )_1 \pm \tan(\ )_2}{1 \mp \tan(\ )_1 \tan(\ )_2}$
k) $\tan(2a) = \dfrac{2 \tan a}{1 - \tan^2 a};$	$\tan(2(\ )) = \dfrac{2 \tan(\ )}{1 - \tan^2(\ )}$

In solving problems involving trigonometric identities, you must be able to recognize the identities when they appear. Therefore we recommend memorizing the basic forms listed above. Since memory can sometimes fail, it may be useful to understand how some of them are related. For example, dividing (a) by $\cos^2 a$ gives (b), while dividing (a) by $\sin^2 a$ implies (c).

The alternate forms of the above identities are really more useful than the basic forms. They allow us to apply the identities in many situations that might not otherwise be apparent.

For example, using the alternative form of identity (e) we can write:

$$\begin{aligned} \sin 6\theta = \sin 2(3\theta) = \sin 2(\ ) &= 2\sin(\ )\cos(\ ) \\ &= 2\sin 3\theta\cos 3\theta, \text{ or} \\ \sin 6\theta &= 2\sin 3\theta \cos 3\theta \end{aligned}$$

Similarly,

$$\begin{aligned} \sin 8\theta =& 2\sin 4\theta\cos 4\theta, \\ \sin 10\theta =& 2\sin 5\theta\cos 5\theta, \text{ etc.} \end{aligned}$$

(Now you see why it's called the "double-angle" formula.)

As another example, consider $\sin 3\theta$. Using the alternate form of identity (d), we have:

$$\begin{aligned} \sin 3\theta &= \sin((2\theta) + (\theta)) = \sin((\ )_1 + (\ )_2) \\ &= \sin(\ )_1\cos(\ )_2 + \cos(\ )_1\sin(\ )_2, \text{ or} \\ \sin 3\theta &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \end{aligned}$$

Remark: The alternate forms of the identities merely use blank parentheses. This idea of using blank parentheses instead of letters or symbols can be extremely beneficial regardless of the course or setting or problem. We strongly recommend that you try to use this idea in all your math classes. (If this remark does not make sense right now, ask your teacher about it when you have a chance. It's that important!)

The basic identities listed earlier along with the definitions of the trigonometric functions (Review Topic 9a) can be used to verify other identities.

For example, verify that

$$\frac{\sin\theta}{1-\sin^2\theta} = \sec\theta\tan\theta.$$

One solution technique is to start with one side and work toward the other using known information. Here, we begin with the right side.

$$\sec\theta\tan\theta = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{1-\sin^2\theta}.$$

A slightly more difficult example is the following. Verify

$$\frac{1+\tan\theta}{1+\cot\theta} = \tan\theta. \tag{14.3}$$

Let's begin with the more complicated expression on the left.

$$\begin{aligned}
\frac{1+\tan\theta}{1+\cot\theta} &= \frac{1+\tan\theta}{1+\dfrac{1}{\tan\theta}} && \text{(definition of} \cot\theta\text{)}\\
&= \frac{1+\tan\theta}{\dfrac{\tan\theta+1}{\tan\theta}} && \text{(adding fractions in the denominator)}\\
&= (1+\tan\theta)\div\left(\frac{\tan\theta+1}{\tan\theta}\right) && \text{(equivalent form)}\\
&= (1+\tan\theta)\cdot\frac{\tan\theta}{(\tan\theta+1)} && \text{(definition of division)}\\
&= \tan\theta.
\end{aligned}$$

In working with identities, sometimes writing everything in terms of $\sin\theta$ or $\cos\theta$ may help. Let's try (14.3) again using this approach.

$$\begin{aligned}
\frac{1+\tan\theta}{1+\cot\theta} &= \frac{1+\dfrac{\sin\theta}{\cos\theta}}{1+\dfrac{\cos\theta}{\sin\theta}} && \text{(definitions)}\\
&= \frac{\dfrac{\cos\theta+\sin\theta}{\cos\theta}}{\dfrac{\sin\theta+\cos\theta}{\sin\theta}} && \text{(adding fractions)}\\
&= \frac{\cos\theta+\sin\theta}{\cos\theta} \div \frac{\sin\theta+\cos\theta}{\sin\theta} && \text{(equivalent form)}\\
&= \frac{(\cos\theta+\sin\theta)}{\cos\theta}\cdot\frac{\sin\theta}{(\sin\theta+\cos\theta)} && \text{(definition of division)}\\
&= \frac{\sin\theta}{\cos\theta} = \tan\theta.
\end{aligned}$$

In calculus, you will not be asked to verify identities. Instead, you will start with an expression and, using identities, manipulate it into an equivalent form more useful for the problem at hand. Two situations where this need will arise involve solving trigonometric equations and performing a process called integration.

PRACTICE PROBLEMS for Topic 14 – Trigonometric Identities
Verify the following identities.

14.1 $\sin\theta\sec\theta = \tan\theta$

14.2 $\sin^2\theta = \tan\theta\cot\theta - \cos^2\theta$

14.3 $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$

14.4 $\sec^2\theta\csc^2\theta = \sec^2\theta + \csc^2\theta$

14.5 $\cos^2 2\theta - \sin^2 2\theta = \cos 4\theta$

ANSWERS to SAMPLE PROBLEMS (Topic 14–Trigonometric Identities)

14.1. $\sin\theta\sec\theta = \sin\theta\cdot\dfrac{1}{\cos\theta} = \tan\theta$

14.2. $\tan\theta\cot\theta - \cos^2\theta = \tan\theta\cdot\dfrac{1}{\tan\theta} - \cos^2\theta = 1-\cos^2\theta = \sin^2\theta$

14.3.

$$\begin{aligned}
\sin\theta(\cot\theta+\tan\theta) &= \sin\theta\left(\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\right)\\
&= \sin\theta\left(\frac{\cos\theta\cos\theta}{\sin\theta\cos\theta}+\frac{\sin\theta\sin\theta}{\cos\theta\sin\theta}\right)\\
&= \sin\theta\left(\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cdot\cos\theta}\right) = \frac{\sin\theta(1)}{\sin\theta\cos\theta}\\
&= \frac{1}{\cos\theta} = \sec\theta
\end{aligned}$$

14.4

$$\begin{aligned}
\sec^2\theta+\csc^2\theta &= \frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}\\
&= \frac{\sin^2\theta}{\cos^2\theta\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}\\
&= \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta}\\
&= \frac{1}{\sin^2\theta}\cdot\frac{1}{\cos^2\theta} = \csc^2\theta\sec^2\theta
\end{aligned}$$

ANSWERS to SAMPLE PROBLEMS (Topic 14–Trigonometric Identities)

14.5 To solve this problem, we first notice that the arguments are not the same. Nearly always, this means the expressions must be manipulated until all the arguments are the same. Typically, we begin with the expressions involving the larger arguments. The idea of using blank parentheses is also crucial here. Based on the alternate form of the first identity in (e), we have

$$\begin{aligned}\cos 4\theta &= \cos 2(2\theta)\\ &= \cos 2(\ ) = \cos^2(\ ) - \sin^2(\ ) = \cos^2(2\theta) - \sin^2(2\theta).\end{aligned}$$

Beginning of Topic Skills Assessment