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MATH 150 – TOPIC 15
INVERSE TRIGONOMETRIC FUNCTIONS

I. This section discusses the inverses of the trigonometric functions. We begin by finding the inverse of the sine function.

From Review Topic 13, we see that $f(x) = \sin x$ has the following graph.

$\sin x$

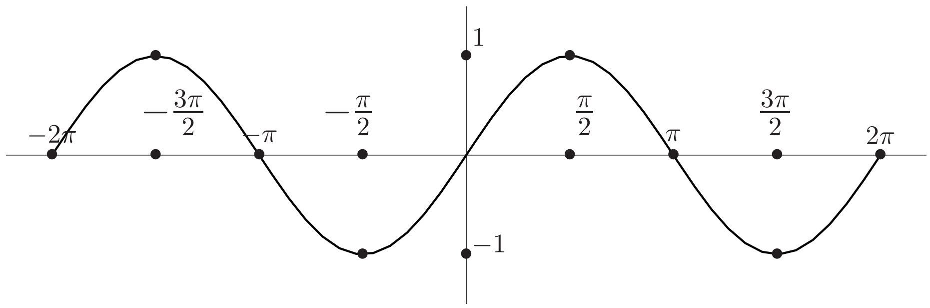


Fig. 15.1.

Remember that in order for a function $f(x)$ to have an inverse, its graph must pass the horizontal line test HTL (a horizontal line intersects the graph at most once). Clearly the graph in Fig. 15.1 fails the HLT. So how do we define the inverse of $\sin x$? Easily. We merely restrict the domain of $\sin x$ to the interval $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$ as in Fig. 15.2 below. On this interval, the graph does pass the HLT and still has range $[-1, 1]$. Thus, $\sin^{-1} x$ or $\arcsin x$ exists.

$\sin x, -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$

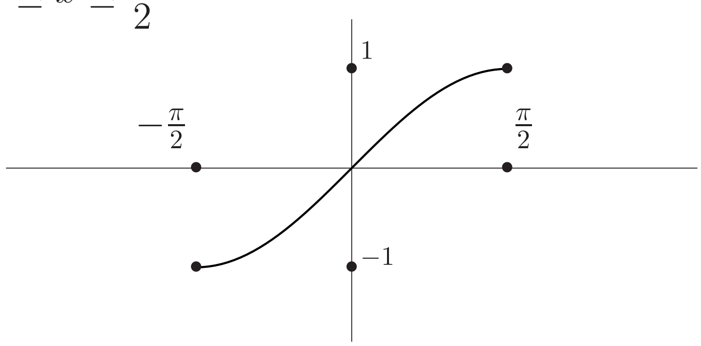
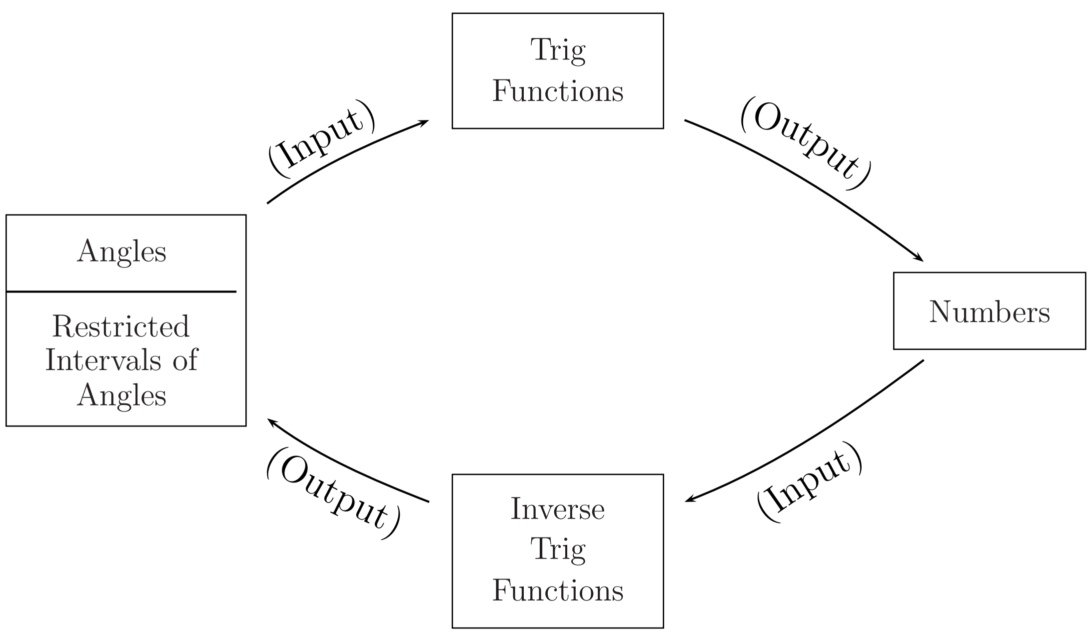


Fig. 15.2.

Let's now discuss some of the properties of $\arcsin(x)$. First, $\sin^{-1}(x)$ or $\arcsin(x)$ "reverses" $\sin x$. That is, if $\sin\left(\frac{\pi}{2}\right) = 1$, then $\sin^{-1}(1)$ or $\arcsin(1) = \frac{\pi}{2}$. (This is true for inverse functions in general. That is, if $f(a) = b$, then $f^{-1}(b) = a$.) Similarly, since $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, then $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

Warning! We know $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, but $\arcsin\left(\frac{1}{2}\right) \neq \frac{5\pi}{6}$!! Why not?! Because $\arcsin\left(\frac{1}{2}\right)$ must be an angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In particular, since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, we conclude $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (not $\frac{5\pi}{6}$). The picture below may help.



Based on the picture above, the sine function has angles as inputs and numbers as outputs, whereas $\arcsin(x)$ has numbers as inputs and **angles** in a restricted interval as outputs. Sometimes this can be helpful in evaluating $\arcsin(x)$. For example, $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ is the **angle** in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

whose sine is $\frac{\sqrt{3}}{2}$. Thus $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

PRACTICE PROBLEMS for Topic 15 – (Math 150 Inverse Trigonometric Functions)

15.1 Evaluate the following without using a calculator. (See Review Topic 12 for exact values of $\sin x$.)

a) $\arcsin\left(\frac{1}{\sqrt{2}}\right)$; b) $\sin^{-1}(1)$; c) $\arcsin(-1)$;

d) $\arcsin(2)$; e) If $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, then $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = ?$

II. This section discusses the graphs of the inverse trigonometric functions.

The graph of $\arcsin x$ or $\sin^{-1}(x)$ can be easily obtained from Fig. 15.2. The domain of $\sin x$ (the set of inputs) is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range (the set of outputs) is $[-1, 1]$. For the inverse function these "flip-flop". Therefore we have the following.

$$\text{For } \arcsin x, \text{ domain} = [-1,1] \text{ and range} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Remember that if (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}. This means that the graph of f^{-1} can be obtained by rotating the graph of f about the line $y = x$. Using this idea of rotation and/or by plotting some points, we see that $\arcsin(x)$ has the following graph.

$\arcsin(x)$ or $\sin^{-1}(x)$

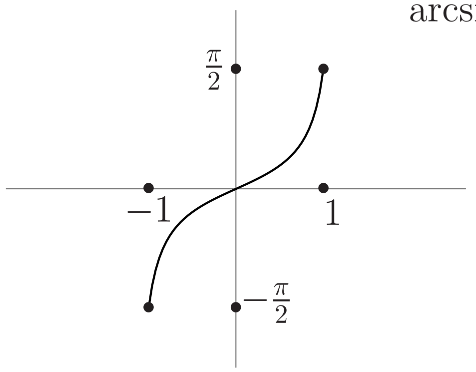


Fig. 15.3.

The derivations of the graphs of $\arccos(x)$ and $\arctan(x)$ are similar to that of $\arcsin(x)$. For $\arccos(x)$, we first consider $\cos x$ where $0 \le x \le \pi$.

$\cos x$, $0 \le x \le \pi$

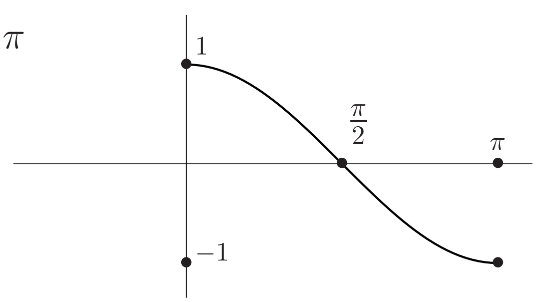


Fig. 15.4.

Since the graph of $\cos x$ passes the HLT on this restricted domain, $\arccos(x)$ or $\cos^{-1}(x)$ exists. Problem 15.2 below investigates $\arccos x$ and its graph.

To derive $\arctan x$, we restrict the domain of $\tan x$ to the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. From Review Topic 13, we see that on this interval the graph of $\tan x$ has the following form.

$\tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

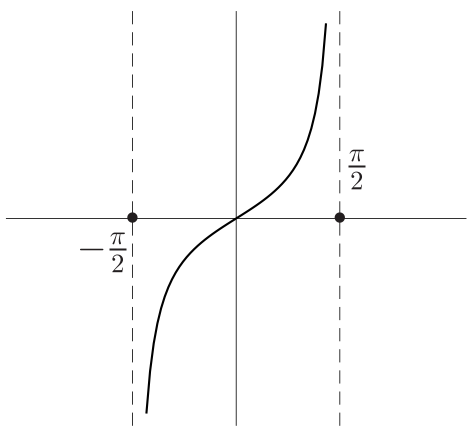


Fig. 15.5.

Once again, the graph in Fig. 15.5 passes the HLT and so $\tan^{-1}(x)$ or $\arctan(x)$ exists.

PRACTICE PROBLEMS for Topic 15 – Inverse Trigonometric Functions

15.2 Without using a calculator:

a) evaluate $\arccos\left(\frac{1}{\sqrt{2}}\right)$, $\arccos(0), \arccos\left(-\frac{1}{2}\right)$.

b) what is the domain of $\arccos(x)$?

c) what is the range of $\arccos(x)$?

d) graph $\arccos(x)$.

15.3 Without using a calculator:

a) evaluate $\arctan(1), \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$, and $\arctan(0)$.

b) find the domain and range of $\arctan(x)$.

c) graph $\arctan(x)$.

III. The idea of composition of functions (Review Topic 3) is also relevant here. For example, how do we interpret $f^{-1}(f(a))$, which is the same as $(f^{-1} \circ f)(a)$? Consider Fig. 15.6 below, where $f(a) = b$ and $f^{-1}(b) = a$.

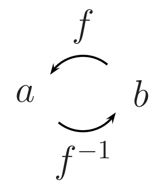


Fig. 15.6.

Intuitively, f and f^{-1} "undo" each other. That is,

$$f^{-1}(f(a)) = a \text{ and } f(f^{(-1)}(b)) = b. \tag{15.1}$$

This means $\sin(\arcsin(.7)) = .7$ and $\arcsin\left(\sin\left(\frac{\pi}{13}\right)\right) = \frac{\pi}{13}$. These answers were obtained not by doing any evaluations but by utilizing the properties illustrated in (15.1).

Warning! You cannot use (15.1) blindly. You must keep in mind the domains and ranges of the functions involved. For example, $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right) \neq \frac{3\pi}{2}$! The statement $\arcsin\left(\quad\right) = \frac{3\pi}{2}$ requires $\frac{3\pi}{2}$ to be an output of $\arcsin(x)$. This is not possible since the range for $\arcsin(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Instead, we write $\arcsin\left(\sin\left(\frac{3\pi}{2}\right)\right) = \arcsin(-1) = -\frac{\pi}{2}$. Problem 15.4d below poses a similar question for $\arccos x$.

PRACTICE PROBLEMS for Topic 15 – Inverse Trigonometric Functions

15.4 Evaluate the following without a calculator.

a) $\cos(\arccos(.7))$

b) $\tan^{-1}\left(\tan\left(-\frac{\pi}{8}\right)\right)$

c) $\sin(\arctan(1))$

d) $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

15.5 If $0 < x < 1$, find $\sin(\arccos(x)$ and $\tan(\arccos(x))$.

15.6 Using a calculator, find:

a) $\tan^{-1}(30.1)$,

b) $\arcsin(-.8)$,

c) $\arccos(.28)$.

IV. ANSWERS to PRACTICE PROBLEMS (Topic 15–Inverse Trigonometric Functions)

15.1. a) $\arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$;

b) $\sin^{-1}(1) = \frac{\pi}{2}$;

c) $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$;

d) arcsin(2) has no solution.
There is no angle whose sine = 2.

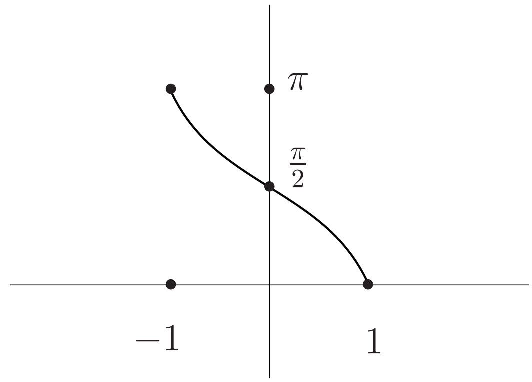
e) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ = angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\left(-\frac{\sqrt{2}}{2}\right)$. So, $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

15.2. a) $\arccos\left(\frac{1}{\sqrt{2}}\right)$ = angle in the interval $[0, \pi]$ whose cosine is $\frac{1}{\sqrt{2}}$.
So, $\arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$. $\arccos(0) = \frac{\pi}{2}$; $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

b) Domain = $[-1, 1]$

c) Range = $[0, \pi]$

d) Graph of $\arccos(x)$

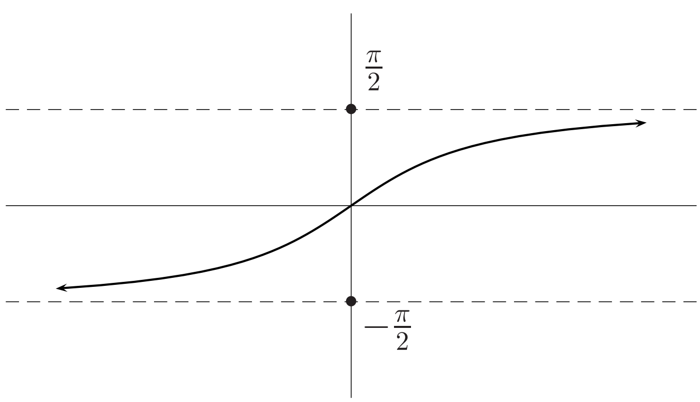


15.3. a) $\arctan(1) = \underline{\text{angle}}\ \underline{\text{in}}\ \underline{\text{the}}\ \underline{\text{interval}}\ \underline{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$ whose tan is 1

$$= \frac{\pi}{4};\ \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}; \quad \arctan(0) = 0.$$

b) Domain $= (-\infty, \infty)$; range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

c) Graph of $\arctan x$



15.4. a) $\cos(\arccos(.7)) = .7$

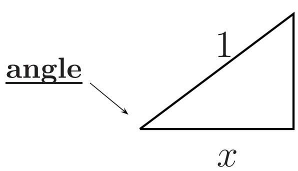
b) $\tan^{-1}\left(\tan\left(\frac{-\pi}{8}\right)\right) = -\frac{\pi}{8}$

c) $\sin(\arctan(1)) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

d) $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$.

Note: $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right) \neq \frac{5\pi}{4}$, since $\frac{5\pi}{4}$ is not in the range of $\cos^{-1}(\ )$ or $\arccos(\ )$.

15.5. These are tricky. Arccos(x) means the **angle** whose cosine $= x = \frac{x}{1}$. This leads to the following picture.



The Pythagorean Theorem implies the missing side has length $\sqrt{1-x^2}$.

Thus, $\sin(\arccos(x)) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$, and

$\tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$.

15.6. a) $\tan^{-1}(30.1) = 1.54$;

b) $\arcsin(-.8) = -.93$;

c) $\arccos(.28) = 1.29$.

Beginning of Topic Skills Assessment