MATH 150 – TOPIC 5

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

I. A log (shorthand for logarithm) is an expression which represents an exponent. Finding the log really means finding the exponent.

Example: The value of $\log_3 9$ is 2 because $3^2 = 9$. The value of $\log_{25} 5$ is $\dfrac{1}{2}$ because $25^{1/2} = 5$.

Even though you are comfortable with exponential forms, much of calculus is written using log notation.

$3^4 = 81$	becomes	$\log_3 81 = 4$	
$10^{-2} = .01$	becomes	$\log .01 = -2$	Base 10 logs are called common logs and the '10' is left out of the notation.

In general, $a^x = b$ in log form becomes $x = \log_a b$ where $a > 0$, $b > 0$, and $a \neq 1$.

II. How do we get e?

Suppose we evaluate $\left(1 + \dfrac{1}{n}\right)^n$ for increasingly larger n.

n	$\left(1 + \dfrac{1}{n}\right)^n$
1	2.0000
10	2.5937
100	2.7048
1,000	2.7169
1,000,000	2.7182

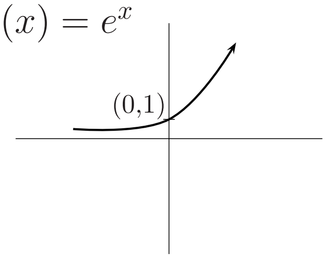
This irrational number is designated by the letter e. That is, $\left(1 + \dfrac{1}{n}\right)^n \to e$ as $n \to \infty$. When writing a log with base e, we use the ln form (natural log).

NOTE: There are other methods to derive e.

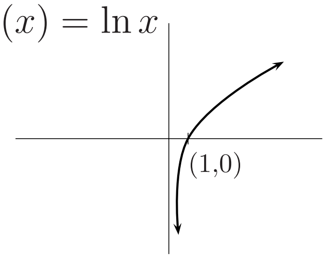
III. e^x and $\ln x$ as functions.

1. Graphs

$f(x) = e^x$

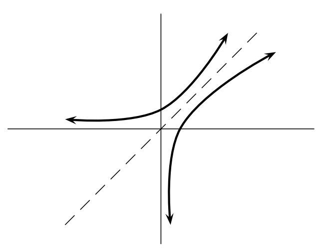


$f(x) = \ln x$



	$\mathbf{y = e^x}$	$\mathbf{y} = \ln \mathbf{x}$
D	all reals	$(0, \infty)$
R	$(0, \infty)$	all reals
	increasing	increasing
	concave up	concave down
asymptotes	horizontal, $y = 0$	vertical, $x = 0$

2. When both functions are plotted on the same axis, do you recognize the symmetry about the line $y = x$?



This means that e^x and $\ln x$ are inverses [for more discussion on inverse functions, refer to Review Topic 15] and leads to two important properties:

$e^{\ln x} = x$ for all $x > 0$. Ex. $e^{\ln \sqrt{2}} = \sqrt{2}$

$\ln e^x = \text{x}$ for all real x. Ex. $\ln e^{\pi} = \pi$

IV. Laws of Logarithms. If M and N are positive, $b > 0$, and $b \neq 1$, then

$$\log_b MN = \log_b M + \log_b N \qquad \text{or} \qquad \ln MN = \ln M + \ln N$$

$$\log_b \frac{M}{N} = \log_b M - \log_b N \qquad \text{or} \qquad \ln \frac{M}{N} = \ln M - \ln N$$

$$\log_b (N^k) = k \log_b N \qquad \text{or} \qquad \ln(N^k) = k \ln N$$

* Don't these look like exponential laws? Why should they?

Here are examples of what these laws can do.

Example: Express $\ln \dfrac{x^2+2}{\sqrt{x}(x-1)}$ in the form involving sums, differences, and multiples of logarithms.

$$\begin{aligned}\ln \frac{x^2+2}{x^{1/2}(x-1)} &= \ln(x^2+2) - \ln[x^{1/2}(x-1)] \\ &= \ln(x^2+2) - [\ln x^{1/2} + \ln(x-1)] \\ &= \ln(x^2+2) - \frac{1}{2}\ln x - \ln(x-1).\end{aligned}$$

Example: Solve for t: $3e^{.2t} = 10$.

Solution:

$$\begin{aligned} e^{.2t} &= \frac{10}{3} \\ \ln e^{.2t} &= \ln \frac{10}{3} \\ .2t &= \ln \frac{10}{3} \\ t &= \frac{\ln \frac{10}{3}}{.2} \qquad \text{or} \qquad t = 5 \ln \frac{10}{3}\end{aligned}$$

Example: Solve for x: $2xe^{2x} + e^{2x} = 0$

Solution: $e^{2x}(2x+1) = 0$

$$x = -\frac{1}{2} \qquad (e^{2x} > 0 \text{ for all } x)$$

PRACTICE PROBLEMS for Topic 5 – Exponential and Logarithmic Functions

5.1. Find the value of each of the following (no calculator, please).

a) $\log 1000$

b) $\log .001$

c) $\log_2 16$

d) $\ln e^{\sqrt{2}}$

e) $\log_{16} 4$

f) $\log_{16} \frac{1}{4}$

g) $\log_{16}(-4)$

h) $\ln 1$

i) $\log_2 10 - \log_2 5$

5.2. Sketch graphs of each of the following. Indicate intercepts and all asymptotes. [Refer to Topic 1 for help on graphing transformations.]

a) $f(x) = e^x$
$g(x) = e^{-x}$
$h(x) = e^{-x} - 2$

b) $f(x) = \ln x$
$g(x) = -\ln x$
$h(x) = \ln(-x)$

5.3. Use laws of logarithms to write $f(x)$ as an expression involving sums, differences, and multiples of natural logarithms.

a) $f(x) = \ln \dfrac{(x-1)(x+3)^2}{\sqrt{x^2+2}}$

b) $f(x) = \ln \dfrac{(x-1)}{\sqrt[3]{x^2+2}\,(x+3)^2}$

5.4. a) Simplify: $\ln(x^2 e^3)$

b) Show how $\dfrac{1}{e^{-x}+1}$ is equivalent to $\dfrac{e^x}{e^x+1}$.

5.5. Solve for x;

a) $e^{3x+5} = 100$

b) $e^{2\ln x} = x + 2$

c) $-3x^2 e^{-3x} + 2xe^{-3x} = 0$

5.6. The number N of bacteria present in a culture at time t (in hours) obeys the equation $N = 1000e^{.02t}$. How long will it take for the population to double?

ANSWERS to PRACTICE PROBLEMS (Topic 5 – Exponential and Logarithmic Functions)

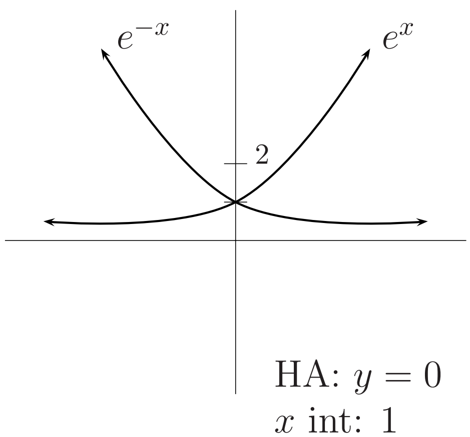
5.1 a) 3 b) -3 c) 4

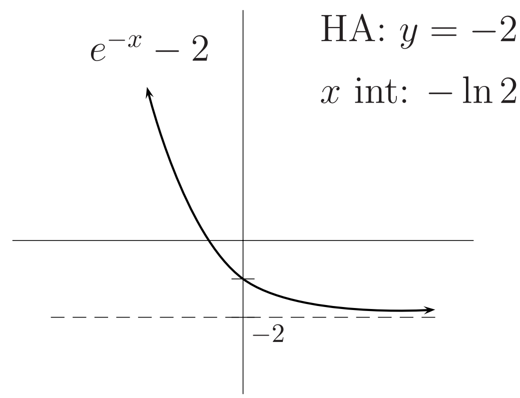
d) $\sqrt{2}$ e) $\dfrac{1}{2}$ f) $-\dfrac{1}{2}$

g) undefined, $\log_a x$ exists only for $x > 0$

h) 0 i) $\log_2 \dfrac{10}{5} = \log_2 2 = 1$

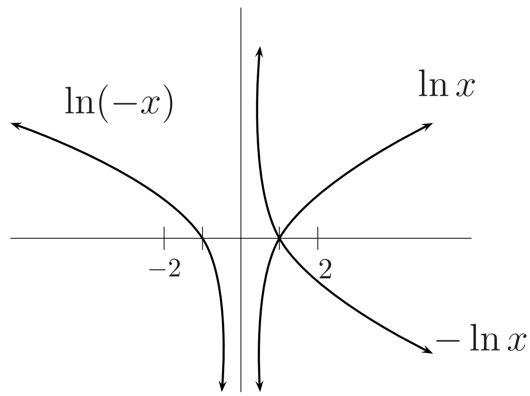
5.2. a)





“HA” means horizontal asymptote. (See Review Topic 6.)

b)



VA: $x = 0$

"VA" means vertical asymptote. (See Review Topic 6.)

5.3. a) $\ln(x-1) + 2\ln(x+3) - \dfrac{1}{2}\ln(x^2+2)$

b) $\ln(x-1) - \dfrac{1}{3}\ln(x^2+2) - 2\ln(x+3)$

5.4. a) $\ln x^2 + \ln e^3 = \ln x^2 + 3$ or $2\ln x + 3$

b) $\dfrac{1}{e^{-x}+1} \cdot \dfrac{e^x}{e^x} = \dfrac{e^x}{e^0 + e^x} = \dfrac{e^x}{1+e^x}$

5.5 a)
$$\ln e^{3x+5} = \ln 100$$
$$3x + 5 = \ln 100$$
$$x = \frac{1}{3}(\ln 100 - 5)$$

b)
$$e^{\ln x^2} = x + 2$$
$$x^2 - x - 2 = 0$$
$$x = 2, x = -1$$
But $x = -1$ cannot be a solution. Why? Because $\ln x$ is not defined if x is negative.

c) $xe^{-3x}(-3x+2) = 0$

$x = 0,\ x = \dfrac{2}{3},\quad (e^{-3x} > 0 \text{ for all } x)$

5.6.
$$\begin{aligned} 2000 &= 1000\, e^{.02t} \\ 2 &= e^{.02t} \\ \ln 2 &= .02t \\ \frac{\ln 2}{.02} &= t \end{aligned}$$

Beginning of Topic Skills Assessment