

MATH 150 – TOPIC 9
RIGHT TRIANGLE TRIGONOMETRY

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9b. Practice Problems

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9c. Practice Problems

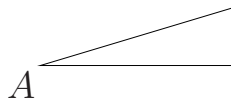
9d. Reference Angles and Trigonometric Functions

9d. Practice Problems

9a. Right Triangle Definitions of the Trigonometric Functions

In this section, we present the definitions of the trigonometric functions based on a right triangle. An alternate but equivalent method of introducing the trigonometric functions is based on the unit circle (Review Topic 11).

Consider some acute angle, A .



Form a right triangle, where $C = \frac{\pi}{2} = 90^\circ$. (Remember that the sum of the interior angles in any triangle is 180° or π radians.)

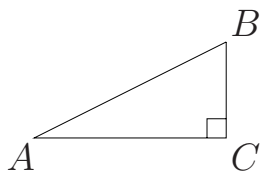


Fig. 9a.1

We have the following definitions.

$$\begin{aligned}\sin A &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AB} \\ \cos A &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AC}{AB} \\ \tan A &= \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin A}{\cos A} = \frac{BC}{AC} \\ \cot A &= \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\tan A} = \frac{\cos A}{\sin A} = \frac{AC}{BC} \\ \sec A &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{\cos A} = \frac{AB}{AC} \\ \csc A &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\sin A} = \frac{AB}{BC}\end{aligned}$$

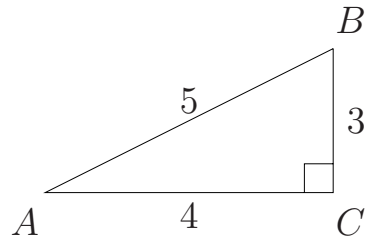
In a similar fashion, we could start with the other acute angle in the right triangle, angle B , and define

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AC}{AB}, \quad \cos B = \frac{BC}{AB}, \text{ etc.}$$

Notice that opposite and adjacent are relative to the angle in question but that the definitions do not change. Also remember that since C is a right angle,

$$B = 90^\circ - A^\circ = \left(\frac{\pi}{2} - A\right) \text{ radians.}$$

PRACTICE PROBLEMS for Topic 9a-Right Triangle Definitions of the Trigonometric Functions



9a.1. Find exact values for the following.

a) $\sin A =$

b) $\cos A =$

c) $\tan A =$

d) $\sin B =$

e) $\cos B =$

f) $\cot B =$

g) $\sec A =$

h) $\sin^2 A + \cos^2 A =$

i) $\csc B =$

j) $\sin\left(\frac{\pi}{2} - A\right) =$

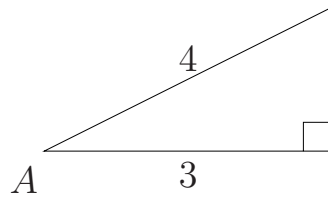
9a.2. Find exact values for all remaining trig ratios given that

$$\sec A = \frac{4}{3}.$$

ANSWERS to PRACTICE PROBLEMS (Topic 9a-Right Triangle Definitions of the Trigonometric Functions)

9a.1. a) $\sin A = \frac{3}{5}$; b) $\cos A = \frac{4}{5}$; c) $\tan A = \frac{3}{4}$; d) $\sin B = \frac{4}{5}$;
e) $\cos B = \frac{3}{5}$; f) $\cot B = \frac{3}{4}$; g) $\sec A = \frac{5}{4}$; h) $\sin^2 A + \cos^2 A = 1$;
i) $\csc B = \frac{5}{4}$; j) $\sin\left(\frac{\pi}{2} - A\right) = \frac{4}{5}$;

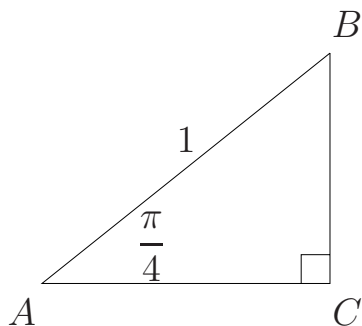
9a.2. $\sec A = \frac{4}{3}$ implies we have the following picture.



The Pythagorean Theorem implies that the missing side must equal $\sqrt{16 - 9} = \sqrt{7}$. Thus, $\sin A = \frac{\sqrt{7}}{4}$, $\cos A = \frac{3}{4}$,
 $\tan A = \frac{\sqrt{7}}{3}$, $\cot A = \frac{3}{\sqrt{7}}$, and $\csc A = \frac{4}{\sqrt{7}}$.

9b. 45°-45°-90° and 30°-60°-90° Triangles Review

The two types of triangles mentioned in the title often appear in calculus. For the acute angles in these triangles, we will find the exact values of the associated trigonometric functions by studying the relationships between the functions, the lengths of the sides, and the angles. We begin with Fig. 9b.1.



Given:

$$AB = 1$$

$$\angle A = \frac{\pi}{4} \text{ rad} = 45^\circ$$

$$\angle C = \frac{\pi}{2} \text{ rad} = 90^\circ.$$

Fig. 9b.1

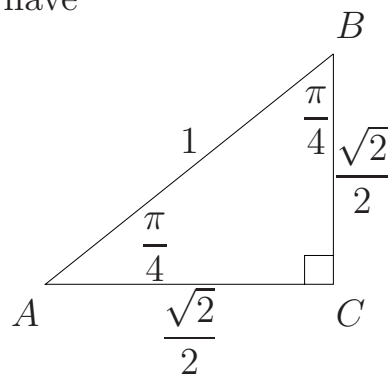
Since the sum of the interior angles is π or 180° , $\angle B$ also = $\frac{\pi}{4}$ radians. This means that the triangle is isosceles, which implies $AC = BC$. By the Pythagorean Theorem,

$$(AC)^2 + (BC)^2 = 1,$$

$$2(BC)^2 = 1,$$

$$BC = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = AC.$$

Thus we have



This implies that $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$,

$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2},$$

$$\tan \frac{\pi}{4} = 1 = \cot \frac{\pi}{4}, \text{ etc.}$$

Fig. 9b.2

Next consider a 30° - 60° - 90° triangle.

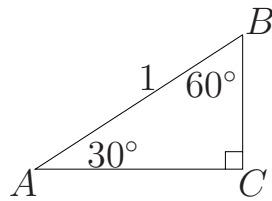


Fig. 9b.3

Given:

$$\angle A = 30^\circ = \frac{\pi}{6} \text{ rad}$$

$$\angle B = 60^\circ = \frac{\pi}{3} \text{ rad}$$

$$\angle C = 90^\circ = \frac{\pi}{2} \text{ rad}$$

$$AB = 1$$

We now form Fig. 9b.4, which comes from flipping the triangle in Fig. 9b.3 about the line AC .

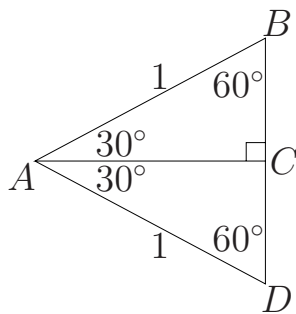


Fig. 9b.4

Evidently, $\triangle ABD$ is an equilateral triangle since each vertex angle is 60° . This means $BD = 1$ and $BC = CD = \frac{1}{2}$, since AC is perpendicular to BD .
(*See remark below.)

So we have

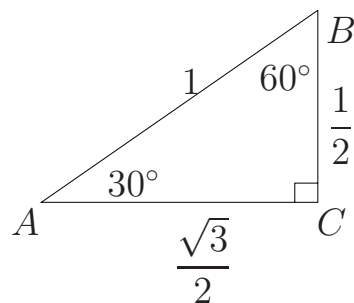


Fig. 9b.5

The Pythagorean Theorem enables us to write $AC = \frac{\sqrt{3}}{2}$. Specifically,
 $1^2 = \left(\frac{1}{2}\right)^2 + (AC)^2$, or $AC = \frac{\sqrt{3}}{2}$. Using right triangle definitions (Review Topic 9a), we see that $\sin 30^\circ = \sin \frac{\pi}{6} = \frac{1}{2}$, $\cos 30^\circ = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$,
 $\sin 60^\circ = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \cos \frac{\pi}{3} = \frac{1}{2}$, $\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$,
 $\tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$, etc.

***Remark:** There is a popular formula that says the side opposite the 30° angle in a right triangle is one half the hypotenuse. Now we see why this is correct.

Based on Figures 9b.2 and 9b.5, we can fill in the following table.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin \theta$			
$\cos \theta$			
$\tan \theta$			

Table 9b.6

PRACTICE PROBLEMS for Topic 9b-45°-45°-90° and 30°-60°-90° Triangles

- 9b.1. Study Figs. 9b.2 and 9b.5 and be able to reconstruct them by memory. Try to understand how the various quantities are derived.
- 9b.2 Fill in Table 9b.6.

- 9b.3 Construct an isosceles right triangle with legs of length 4. Find the sine, cosine, and tangent of $\frac{\pi}{4}$ and compare these values to those found in Fig. 9b.2. What conclusions can be drawn?

ANSWERS to PRACTICE PROBLEMS (Topic 9b–45°–45°–90° and 30°–60°–90° Triangles – Answers)

9b.2

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin \theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan \theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Answer 9b.2

9b.3 The values of the trigonometric functions in Fig. 9b.2 will be the same for any isosceles right triangle.

9c. Determining Lengths of Sides

Right triangle trigonometry can also give the lengths of the sides of a right triangle, if you know at least one of the acute angles and one side of the triangle. For example, in the figure below, find BC and AC .

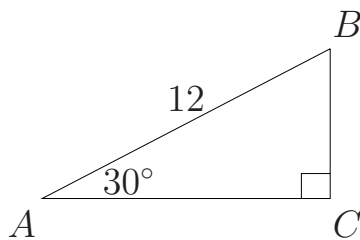


Fig. 9c.1

Since $\sin 30^\circ = \sin \frac{\pi}{6} = \frac{1}{2}$ (see Review Topic 9b), we can write

$\frac{1}{2} = \sin A = \frac{BC}{12}$, or $BC = 6$. Then AC can be obtained by using the cosine or the tangent function (or the Pythagorean Theorem). For example,

$\frac{1}{\sqrt{3}} = \tan A = \frac{BC}{AC}$. Thus $\frac{1}{\sqrt{3}} = \frac{6}{AC}$, or $AC = 6\sqrt{3}$.

Note that since $A = 30^\circ$ (and $C = 90^\circ$), angle $B = 60^\circ$. Then we could apply the same reasoning to angle B and obtain the lengths of the respective sides.

Let's do another example where you have to use a calculator. Consider Fig. 9c.2, where angle $B = 50^\circ$ and $AC = 4$. Find the lengths of AB and BC .

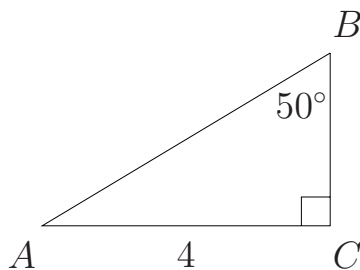


Fig. 9c.2

Using a calculator (set in degree mode!) $\sin 50^\circ = .766$. Thus,

$$.766 = \sin 50^\circ = \sin B = \frac{4}{AB}, \text{ or}$$

$$AB = \frac{4}{.766} = 5.22 \text{ (approximately).}$$

Next,

$$\tan B = \tan 50^\circ = 1.19 = \frac{4}{BC}, \text{ or}$$

$$BC = \frac{4}{1.19} = 3.36.$$

In calculus, sometimes the following problem is encountered. If $\sin \theta = x$, and θ is in the first quadrant, find $\cos \theta$ and $\tan \theta$. To solve this problem, we first note that $x = \frac{x}{1}$. This allows us to create the following picture, since $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$.

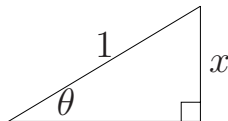


Fig. 9c.3

By the Pythagorean Theorem, we can solve for the missing side (call it a), at least algebraically. That is, $c^2 = a^2 + b^2$, or $1 = a^2 + x^2$, or $a = \sqrt{1 - x^2}$. So we now have the following.

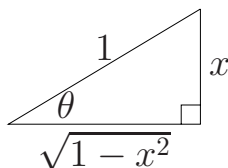


Fig. 9c.4

This means $\cos \theta = \sqrt{1 - x^2}$, and $\tan \theta = \frac{x}{\sqrt{1 - x^2}}$.

PRACTICE PROBLEMS for Topic 9c–Determining Lengths of Sides

9c.1.

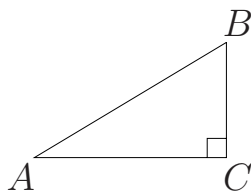


Fig. 9c.5

Suppose $A = 45^\circ = \frac{\pi}{4}$ radians. If $BC = 4$, find AC and AB .

9c.2.

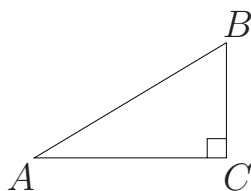


Fig. 9c.6

Suppose $AB = 8$ and angle $A = 1$ radian. Find AC and BC .

9c.3. If $\cos \theta = \frac{x}{2}$ and θ is in the first quadrant, find $\sin \theta$ and $\tan \theta$.

ANSWERS to PRACTICE PROBLEMS (Topic 9c–Determining Lengths of Sides)

9c.1. $\tan \frac{\pi}{4} = 1$. Also, $\tan \frac{\pi}{4} = \frac{BC}{AC}$. Thus, $\frac{BC}{AC} = 1 \Rightarrow AC = 4$. Note that since the triangle is isosceles, elementary geometry also implies that $AC = 4$. Next $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{BC}{AB} = \frac{4}{AB}$. Thus $AB = 4\sqrt{2}$.

(**Reminder:** We can also find AB by using the Pythagorean Theorem.)

9c.2. Using a calculator set in radian mode, we find that $\sin(1) = .84$. This means $\frac{BC}{8} = .84$, or $BC = 6.73$. Next, $\cos(1) = .54$. Since $\cos 1 = \frac{AC}{AB} = \frac{AC}{8}$, we have $AC = 8(.54) = 4.32$.

9c.3. Given $\cos \theta = \frac{x}{2}$ we can draw the following picture.

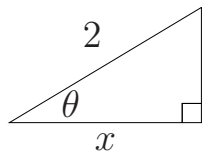


Fig. 9c.7

The Pythagorean Theorem implies the missing side has length $\sqrt{4 - x^2}$.

Then,

$$\sin \theta = \frac{\sqrt{4 - x^2}}{2}, \quad \text{and} \quad \tan \theta = \frac{\sqrt{4 - x^2}}{x}.$$

9d. Reference Angles and the Trigonometric Functions

The definitions of the trigonometric functions in Review Topic 9a were based on having an acute angle (less than 90°). What happens if $A > 90^\circ = \frac{\pi}{2}$ radians? For example, Let A be the angle pictured in Fig. 9d.1. Now, what is meant by $\sin A$, $\cos A$, etc.?

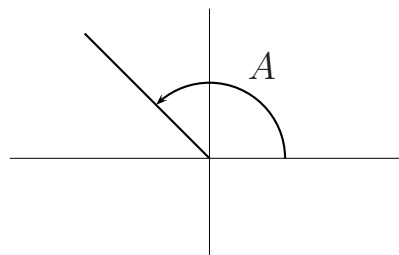


Fig. 9d.1

To answer this question, we consider the reference angle of A , which is the acute angle between the terminal side of A and the x -axis. In this case, the reference angle is $(\pi - A)$ as in Fig. 9d.2. (For the rest of our discussion, the reference angle of A will also be denoted by $\text{ref } A$.)

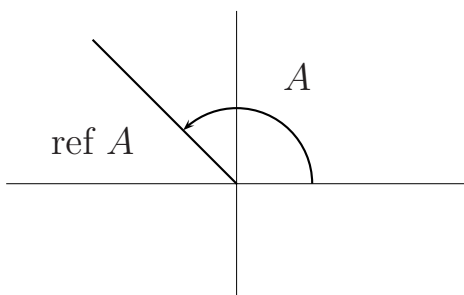


Fig. 9d.2

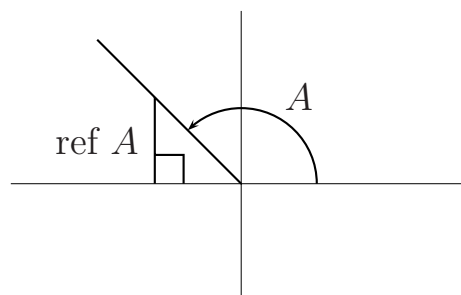


Fig. 9d.3

We next form a right triangle between the terminal side of A and the x -axis, as indicated in Fig. 9d.3. Then we define

$$\sin A = \sin(\text{ref angle}).$$

Note that since $(\text{ref } A)$ is an acute angle with its own right triangle, we can again use definitions involving an opposite side, adjacent side, etc., just as we did for Fig. 9a.1 in Review Topic 9a.

For example, consider Fig. 9d.4 below.

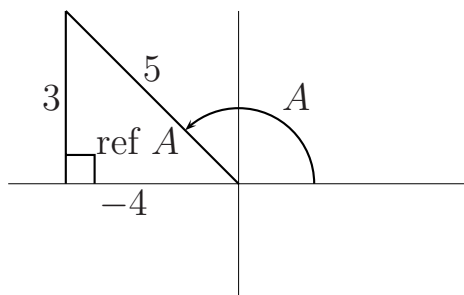


Fig. 9d.4

Then we define $\sin A = \sin(\text{reference angle}) = \frac{3}{5}$, $\cos A = -\frac{4}{5}$,

$\tan A = \frac{3}{-4} = -\frac{3}{4}$, etc. Observe that the adjacent side of the reference angle is (-4) , since x is negative. Also note that since we have a right triangle, the numbers 3, -4 , and 5 satisfy the Pythagorean Theorem.

The situation is similar if the terminal side of A lies in the third or fourth quadrant, as in Fig. 9d.5.

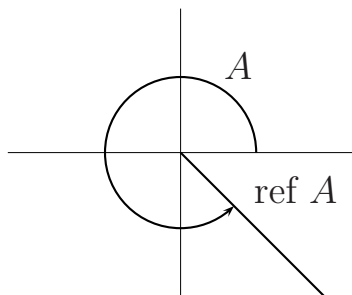


Fig. 9d.5

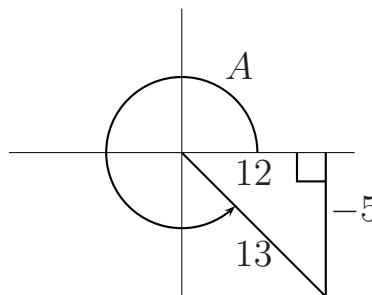


Fig. 9d.6

We form a right triangle with the reference angle as indicated in Fig. 9d.6, and we suppose the lengths of the sides are as shown. Then $\sin A = \sin(\text{reference angle}) = \frac{-5}{13}$, $\cos A = \frac{12}{13}$, $\tan A = \frac{-5}{13}$, etc.

PRACTICE PROBLEMS for Topic 9d-Reference Angles and the Trigonometric Functions

- 9d.1. Consider angle A as in Fig. 9d.7, and suppose its reference angle forms a right triangle with the dimensions shown in Fig. 9d.8. Find $\sin A$, $\cos A$, $\tan A$.

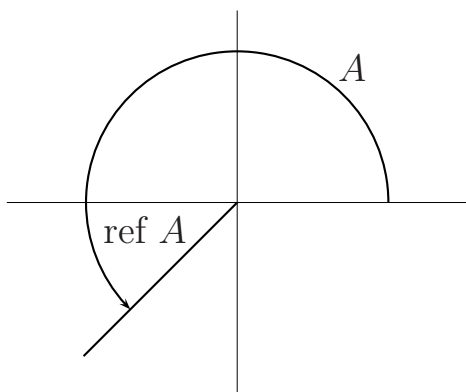


Fig. 9d.7

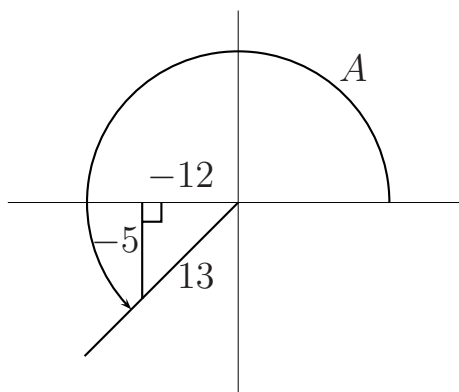


Fig. 9d.8

Evidently, based on Figures 9.d4, 9d.6, and 9d.8, we can conclude that $\sin \theta > 0$ for θ in the first and second quadrants and $\sin \theta < 0$ for θ in the third and fourth quadrants. Similarly, $\cos \theta$ is positive in the first and fourth quadrants and negative in the second and third quadrants, while $\tan \theta$ is positive in the first and third quadrants and negative in the second and fourth quadrants. The table below summarizes these observations.

	1st Q.	2nd Q.	3rd Q.	4th Q.
$\sin \theta$	+	+	-	-
$\cos \theta$	+	-	-	+
$\tan \theta$	+	-	+	-

Table 9d.9 - "Q" means quadrant.

PRACTICE PROBLEMS for Topic 9d-Continued

9d.2. Determine whether the following expressions are positive or negative.

a) $\tan \frac{7\pi}{6}$, b) $\sec \frac{3\pi}{4}$, c) $\sin \frac{11\pi}{6}$, d) $\sin \frac{5\pi}{6}$,

e) $\cos \frac{5\pi}{4}$, f) $\tan \theta$, if $\frac{3\pi}{2} < \theta < 2\pi$.

9d.3. Name the quadrant(s) where θ must terminate, given the following information.

a) $\sin \theta < 0$

b) $\sin \theta > 0$ and $\tan \theta < 0$.

Finally, we can use the idea of reference angles to extend what we know about $45^\circ - 45^\circ - 90^\circ$ and $30^\circ - 60^\circ - 90^\circ$ triangles from the first quadrant to the other quadrants. For example, let $A = \frac{\pi}{6}$. Let B , C , and D be the positive angles in the second, third, and fourth quadrants respectively that have reference angle $\frac{\pi}{6}$. A moment's reflection yields that $B = \frac{5\pi}{6}$, $C = \frac{7\pi}{6}$, and $D = \frac{11\pi}{6}$.

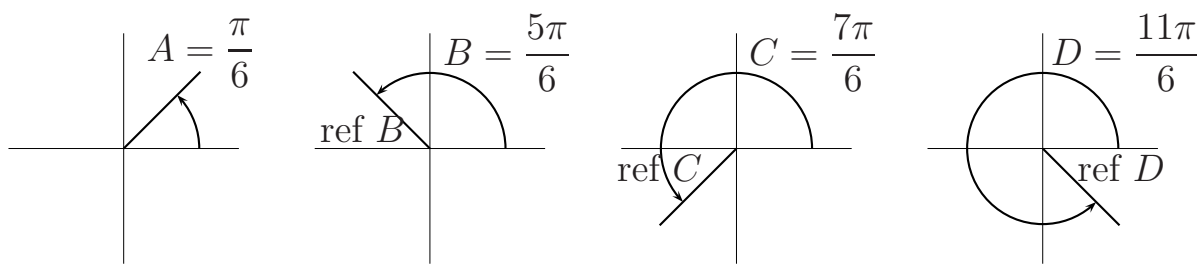


Fig. 9d.10. All reference angles above $= \frac{\pi}{6}$.

As shown in Review Topic 9b, $\sin \frac{\pi}{6} = \frac{1}{2}$. This leads to Fig. 9d.11.

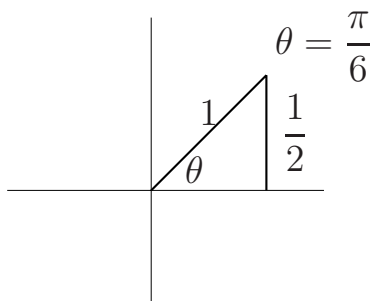


Fig. 9d.11

Figs. 9d.10 and 9d.11 immediately yield the following information.

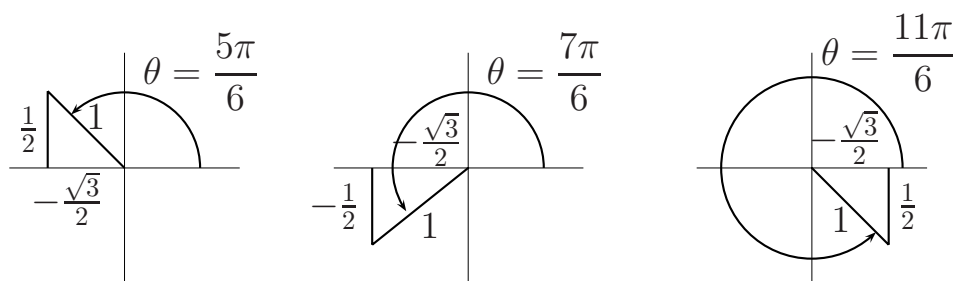


Fig. 9d.12.

Fig. 9d.12 enables us to create the table below.

θ	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$
$\sin \theta$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\cos \theta$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$\tan \theta$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

Table 9d.13

We remark that similar reasoning applies for negative angles. For example, if $\theta = -\frac{\pi}{6}$, it has the same reference angle as the drawing on the right in Fig. 9d.12 (when $\theta = \frac{11\pi}{6}$). Thus, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, etc. As an additional example, $\theta = -\frac{3\pi}{4}$ would have the same reference angle as $\theta = \frac{5\pi}{4}$.

Another observation is that $\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$ and $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$. We will see in Review Topic 13 that this observation is true in general. That is, for any x , $\sin(-x) = -\sin x$ ($\sin x$ is an odd function), $\cos(-x) = \cos x$ ($\cos x$ is an even function), and $\tan(-x) = -\tan x$ ($\tan x$ is an odd function). The concept of even and odd functions is explained in Review Topic 1.

PRACTICE PROBLEMS for Topic 9d-Continued

- 9d.4. What are the positive angles in the second, third, and fourth quadrants that have $\frac{\pi}{3}$ as their reference angle?
- 9d.5. What are the positive angles in the second, third, and fourth quadrants that have $\frac{\pi}{4}$ as their reference angle?

9d.6. Arguing as we did in Fig. 9d.10, 9d.11, and 9d.12, complete the tables below.

θ	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
$\sin \theta$			
$\cos \theta$			
$\tan \theta$			

Table 9d.14

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
$\sin \theta$			
$\cos \theta$			
$\tan \theta$			

Table 9d.15

9d.7. Find the exact value for each expression.

a) $\cos\left(-\frac{\pi}{3}\right)$ b) $\tan\left(-\frac{\pi}{4}\right)$ c) $\sin\left(-\frac{3\pi}{4}\right)$

ANSWERS to PRACTICE PROBLEMS (Topic 9d-Reference Angles and the Trigonometric Functions)

$$9d.1. \quad \sin A = -\frac{5}{13}, \cos A = -\frac{12}{13}, \tan A = \frac{-5}{-12} = \frac{5}{12}.$$

$$9d.2. \quad \text{a) positive} \quad \text{b) negative, since } \sec \theta = \frac{1}{\cos \theta}.$$

$$\text{c) negative} \quad \text{d) positive} \quad \text{e) negative} \quad \text{f) negative}$$

$$9d.3. \quad \text{a) 3rd quadrant or 4th quadrant}$$

$$\text{b) 2nd quadrant}$$

$$9d.4. \quad \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$9d.5. \quad \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$9d.6.$$

θ	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
$\sin \theta$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$\cos \theta$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\tan \theta$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$

Table 9d.15

θ	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
$\sin \theta$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\cos \theta$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\tan \theta$	-1	1	-1

Table 9d.15

9d.7. a) $\frac{1}{2}$, b) -1 c) $-\frac{\sqrt{2}}{2}$.

NOTE: The solution for part a) can be derived in two ways.

First, if we use the concept of reference angle, then $\theta = -\frac{\pi}{3}$ has reference angle $\frac{\pi}{3}$. From Table 9b.6, $\cos \frac{\pi}{3} = \frac{1}{2}$. Since $-\frac{\pi}{3}$ is in the fourth quadrant where the x coordinate is positive, we have $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$.

As a second approach, since $\cos x$ is an even function, (i.e., $\cos(-x) = \cos x$), we can simply say $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. Similar comments apply for answers b) and c).

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