MATH 250 – TOPIC 11

LIMITS

A. Basic Idea of a Limit and Limit Laws

In Calc I, you undoubtedly had problems similar to the following.

$$\text{Let } f(x) = x^2 + 2. \text{ Find } \lim_{x\to 1} f(x).$$

If you said the answer is 3, you are correct. How did you get it? You probably plugged in 1 for x! However, x is never supposed to equal 1. Instead, you are really interested in what $f(x)$ is doing as x gets closer and closer to 1. Let's review what is happening in this limit.

A limit typically has the form

$$\lim_{x\to a} f(x).$$

Notice that there are <u>two</u> things "moving or changing." The input variable x is moving closer to a. As the input x changes, the output $f(x)$ also changes. So now you have a bunch of outputs $f(x)$. Are they getting "close" to some value? You <u>know</u> x is moving arbitrarily close (but never equal) to a. The question is, are the outputs $f(x)$ moving arbitrarily close to some number? Do you <u>expect</u> the outputs to "arrive" at some value? If you do, then this value is the limit.

Once again. You <u>know</u> where x is going. Can you say without a doubt <u>where</u> <u>you</u> <u>expect</u> $f(x)$ to go? If you can, then this value is the limit.

Example 11A.1. Find $\lim_{x\to 1}(2x+3)$.

As x gets close to 1 (written $x \to 1$) from the right side or the left side, we <u>expect</u> $(2x+3)$ to get close to 5. To see that this is really happening, consider the following table.

x	0	.5	.9	.99	.999		1.001	1.01	1.1	1.5	2
$f(x)$	3	4	4.8	4.98	4.998		5.002	5.02	5.2	6	7

Table 11A.1

We see that the closer x gets to 1, the closer $f(x)$ gets to 5. Thus, $\lim_{x\to 1}(2x+3) = 5$.

Notice in Table 11A.1 that we substituted values for x on both sides of 1. Remember:

(i) $\lim\limits_{x\to 1}(\quad)$ means x is allowed to assume values on either side of 1.

(ii) $\lim\limits_{x\to 1^+}(\quad)$ means x is always greater than 1 and is coming down to 1 from the right.

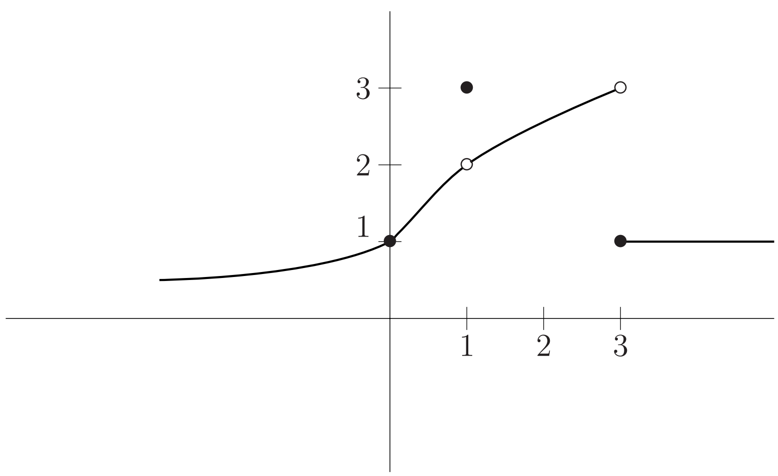
(iii) $\lim\limits_{x\to 1^-}(\quad)$ means x is always less than 1 and is coming up to 1 from the left.

Important Fact from Calc I:

$\lim\limits_{x\to a} f(x)$ exists if and only if $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ both exist and are the same.

This means that if $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$ or if one of these limits doesn't exist, then $\lim\limits_{x\to a} f(x)$ doesn't exist.

Example 11A.2. Consider the following graph.



When the input x moves along the x-axis, the function $f(x)$ "travels" along the graph. As x gets close to 1, you <u>expect</u> that the $f(x)$ or y values on the graph will tend to 2. So, $\lim\limits_{x\to 1} f(x) = 2$. Of course, $f(1) = 3$, which $\neq \lim\limits_{x\to 1} f(x)$. However, this is irrelevant as far as the limit is concerned. The limit is the value you <u>expect</u> $f(x)$ to "hit."

Aside: The fact that $\lim_{x\to 1} f(x) \neq f(1)$ is extremely relevant for the concept of continuity. *Are any bells ringing*?

Practice Problem 11A.1. In Example 11A.2, find:

a) $\lim_{x\to 0} f(x)$; (b) $\lim_{x\to 3^+} f(x)$; (c) $\lim_{x\to 3^-} f(x)$; (d) $\lim_{x\to 3} f(x)$ Answers

As you probably remember, there are limit laws from Calc I.

Example. Find $\lim_{x\to\frac{\pi}{2}}(x+\sin x)$.

Answer: $\frac{\pi}{2}+\sin\left(\frac{\pi}{2}\right)=\frac{\pi}{2}+1\,.$

To do this problem, you (maybe subconsciously!) used the fact that

$$\lim_{x\to\frac{\pi}{2}}(x+\sin x)=\lim_{x\to\frac{\pi}{2}} x+\lim_{x\to\frac{\pi}{2}}\sin x.$$

In general, we have the following rules from Calc I. Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then:

$$\lim_{x\to a}(f(x)\pm g(x))=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x) \tag{11A.1}$$

$$\lim_{x\to a}(cf(x))=c\lim_{x\to a} f(x),\quad \text{for any constant } c. \tag{11A.2}$$

$$\lim_{x\to a}(f(x)g(x))=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) \tag{11A.3}$$

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},\quad \text{provided } \lim_{x\to a} g(x)\neq 0. \tag{11A.4}$$

Use the above rules to evaluate the following limits.

Practice Problem 11A.2. Evaluate:

a) $\lim_{x\to 1}\frac{x+1}{x^2+2}$; (b) $\lim_{x\to\frac{\pi}{4}}\frac{\sin x}{x+\cos x}$; (c) $\lim_{x\to 2}(7-x)\ln x$ Answers

B. Limits of the form $\dfrac{\mathbf{0}}{\neq \mathbf{0}}, \dfrac{\neq \mathbf{0}}{\mathbf{0}}, \dfrac{\mathbf{0}}{\mathbf{0}}$

Example 11B.1. Let $f(x) = x^2 - x$. Find $\lim_{x \to 1} f(x)$.

The answer is 0. You <u>expect</u> the output values of $f(x)$ to get arbitrarily close to 0. Is there anything wrong with getting 0? No! Is this a valid answer? Certainly!

Example 11B.2. Let $f(x) = \dfrac{x^2 - x}{x + 1}$. Find $\lim_{x \to 1} f(x)$.

The answer is still 0, since now you expect that the values of $f(x)$ will tend to $\dfrac{0}{2}$, which is 0.

However, consider the following example.

Example 11B.3. $\lim_{x \to 1} \dfrac{1}{x - 1}$.

As x gets close to 1, your first thought is that the limit is getting close to $\dfrac{1}{0}$, which is undefined! What happens now? Consider another example.

Example 11B.4. Find $\lim_{x \to 0} \dfrac{\sin x}{x}$.

As $x \to 0$, your first thought is that the limit is $\dfrac{0}{0}$. How do we handle this?

Notice that the limits in Examples 11B.2, 11B.3, and 11B.4 have the form $\dfrac{0}{\neq 0}$, $\dfrac{\neq 0}{0}$, and $\dfrac{0}{0}$, respectively. Let us consider each form separately.

Form $\dfrac{\mathbf{0}}{\neq \mathbf{0}}$

Like Example 11B.2, limits having the first form $\dfrac{0}{\neq 0}$ are easy. The limit is zero.

Example 11B.5. Find $\displaystyle\lim_{x\to\pi} \frac{\sin x}{\cos^2 x}$.

As $x \to \pi$, $\dfrac{\sin x}{\cos^2 x} \to \dfrac{0}{1}$ (form $\dfrac{0}{\neq 0}$), which is 0.

Exercise B.1. Evaluate the following limits.

a) $\displaystyle\lim_{x\to 2} \frac{2x^2 - 3x - 2}{x^2 - 2}$

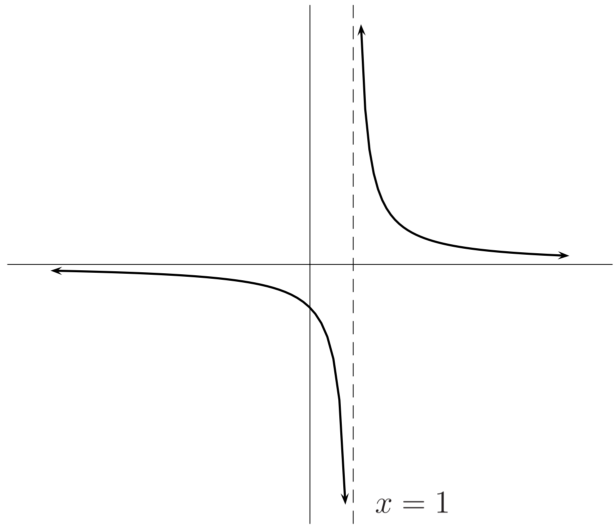
b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\ln(x+2)}$

Answers

Form $\dfrac{\neq \mathbf{0}}{\mathbf{0}}$

Next let's analyze Example 11B.3, where the limit had the form $\dfrac{1}{0}$ ($\dfrac{\neq 0}{0}$). The function in this example is $f(x) = \dfrac{1}{x-1}$. The input $x = 1$ makes the bottom zero <u>but</u> <u>not</u> <u>the</u> <u>top</u> (form $\dfrac{\neq 0}{0}$).

This means the function has a vertical asymptote at $x = 1$. (See Math 150, Review Topic 6, for an explanation of vertical asymptotes.) Thus, as x gets close to 1 from the right or the left, the outputs will become unbounded. (This <u>always</u> happens at a vertical asymptote!). To see what the limits are from either side, we draw a quick sketch of $f(x) = \dfrac{1}{x-1}$ around $x = 1$.



Thus, $\displaystyle\lim_{x\to 1^+} \frac{1}{x-1} = \infty$, and $\displaystyle\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$. Since the limits from the right and left are different, $\displaystyle\lim_{x\to 1} \frac{1}{x-1}$ does not exist.

Remember! Example 11B.3 is typical of limit problems $\displaystyle\lim_{x\to a} f(x)$ <u>which have the form</u> $\dfrac{\neq 0}{0}$. The answer is either ∞, $-\infty$, or does not exist. Since $x = a$ is a vertical asymptote, as $x \to a$ from either side the outputs <u>will</u> become unbounded. You must check both sides to see if the outputs are positive (heading toward ∞) or negative (heading toward $-\infty$).

Example 11B.6. Find $\displaystyle\lim_{x\to \frac{\pi}{2}^-} 2\tan x$.

The function $2\tan x = \dfrac{2\sin x}{\cos x}$. At $x = \dfrac{\pi}{2}$, this has the form $\dfrac{2\cdot 1}{0} = \dfrac{2}{0}$ (form $\dfrac{\neq 0}{0}$). Therefore $x = \dfrac{\pi}{2}$ is a vertical asymptote for $2\tan x$, which means the function becomes unbounded as x approaches $\dfrac{\pi}{2}$ from the left. To see if the limit is $+\infty$ or $-\infty$, consider values of x slightly less than $\dfrac{\pi}{2}$. The sine will be positive and the cosine will also be positive. Therefore $2\tan x$ will be positive. Thus, $\displaystyle\lim_{x\to \frac{\pi}{2}^-} 2\tan x = \infty$.

Note: Example 11B.6 also follows from looking at the graph of the tangent function.

Exercise B.2. Evaluate the following limits.

a) $\displaystyle\lim_{x\to 2^-} \frac{x}{x-2}$

b) $\displaystyle\lim_{x\to \frac{\pi}{3}} \frac{\sin x}{2\cos x - 1}$

Answers

Form $\frac{\mathbf{0}}{\mathbf{0}}$

When a limit has the third form $\frac{0}{0}$, like Example 11B.4, this <u>always</u> means more work. You can't conclude anything as the problem stands, and any answer is possible. You'll have to do some manipulation (or pull some other trick out of your math hat!) to change the starting expression into an equivalent form that lets you see what is happening. We will refer you to your calculus book to read the steps involved in showing $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Example 11B.7. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$ (See your calculus book for details.)

Example 11B.8. Find $\displaystyle\lim_{x\to 4}\left(\frac{\sqrt{x}-2}{x-4}\right)$.

This has form $\frac{0}{0}$, so more manipulation is needed. Here we multiply top and bottom by $\sqrt{x}+2$. We are performing legal algebra and changing the starting expression into an equivalent one that lets us see what is happening.

$$\frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}\,.$$

So, $\displaystyle\lim_{x\to 4}\left(\frac{\sqrt{x}-2}{x-4}\right) = \lim_{x\to 4}\left(\frac{1}{\sqrt{x}+2}\right) = \frac{1}{4}\,.$

Example 11B.9. Find $\lim\limits_{h\to 0}[\frac{f(x+h)-f(x)}{h}]$, where $f(x) = x^2$. Well, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$. This means

$$\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{x\to 4}\left[\frac{x^2+2xh+h^2-x^2}{h}\right].$$

This has the form $\frac{0}{0}$. Here though, the required algebraic manipulation is straightforward.

$$\begin{aligned}\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] &= \lim_{h\to 0}\left[\frac{2xh+h^2}{h}\right] = \lim_{h\to 0}\left[\frac{h(2x+h)}{h}\right]\\ &= \lim_{h\to 0}[2x+h] = 2x.\end{aligned}$$

Note: Taking the limit in any difference quotient as it stands will yield the form $\frac{0}{0}$. Manipulating it to find the limit is the same as calculating the derivative of f using the formal definition of a derivative (Review Topic 12A).

Practice Problems. Find the following limits.

11B.1. $\lim\limits_{x\to 1}\dfrac{\sin x}{\cos^2 x}$ Answer

11B.2. $\lim\limits_{x\to 1}\dfrac{x^2-1}{1-x}$ Answer

11B.3. $\lim\limits_{x\to 2}\dfrac{x+2}{x-2}$ Answer

11B.4. $\lim\limits_{x\to 3}\dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$ Answer

11B.5. $\lim\limits_{x\to 0}\dfrac{\cos x-1}{x}$ Hint: multiply by $(\frac{\cos x+1}{\cos x+1})$. Answer

11B.6. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$, where $f(x) = \frac{1}{x}$. Answer

C. Limits as $\mathbf{x} \to \infty$ or as $\mathbf{x} \to -\infty$

In many situations in Calc II, you will have to evaluate limits where $x \to \infty$ or $x \to -\infty$. Let's do some examples.

Example 11C.1. Find $\lim_{x\to\infty} f(x)$, where $f(x) = x^2 + x$.

Remember our intuitive definition of a limit. We know x is getting large. Is there anything we can say about what we expect for $f(x)$? Yes. Since $f(x) = x^2 + x$, we expect $f(x)$ to get large as x gets large. Thus, $\lim_{x\to\infty} f(x) = \infty$.

Sometimes when processes compete for control, a limit may appear indeterminable.

Example 11C.2. Find $\lim_{x\to\infty} (x^2 - x^3)$.

This problem has the form $\infty - \infty$. There are two competing processes going on, so to speak. Which one wins? Does the first override the second, does the second override the first, or do they offset each other in some fashion? Once again we must do more work or manipulation to see what is happening (and any answer is possible!). We proceed as follows.

$$\lim_{x\to\infty} (x^2 - x^3) = \lim_{x\to\infty} x^2(1 - x).$$

Now we have a product of an arbitrarily large positive number and an arbitrarily large negative number (since $(1 - x)$ is negative). We expect that $x^2(1 - x) \to -\infty$, which is the answer. Thus,

$$\lim_{x\to\infty} (x^2 - x^3) = \lim_{x\to\infty} x^2(1 - x) = -\infty.$$

Example 11C.3. Find $\lim_{x\to\infty} \dfrac{x^2 + 1}{2x^2 - 3}$.

This has the form $\dfrac{\infty}{\infty}$ which (like $\dfrac{0}{0}$) really means two competing processes. Does the numerator win or does the denominator, or do they "counterbalance" each other in some way? Again, we must manipulate the expression to see what is happening. The following procedure is a good tool for problems of this type.

> In limit problems involving quotients of powers of x <u>where</u> $\underline{x \to \infty}$ <u>or</u> $\underline{x \to -\infty}$, factor out the highest power of x in the numerator and the highest power of x in the denominator. (The powers of x may <u>not</u> be the same.)

Doing this for Example 11C.3 yields:

$$\lim_{x\to\infty} \frac{x^2+1}{2x^2-3} = \lim_{x\to\infty} \frac{x^2(1+\frac{1}{x^2})}{x^2(2-\frac{3}{x^2})} = \lim_{x\to\infty} \frac{(1+\frac{1}{x^2})}{(2-\frac{3}{x^2})} = \frac{1}{2}\,.$$

Note: $\frac{1}{x^2}$ and $\frac{3}{x^2}$ both $\to 0$ as $x \to \infty$.

Factoring is also useful in problems like the following (which you will encounter when studying infinite series).

Example 11C.4. Find $\lim\limits_{n\to\infty} [\frac{2^n}{2^n-1}]$.

Once again, this has the form $\frac{\infty}{\infty}$ which means more manipulation is needed to determine the outcome.

$$\lim_{n\to\infty} (\frac{2^n}{2^n-1}) = \lim_{n\to\infty} \left(\frac{2^n \cdot 1}{2^n(1-\frac{1}{2^n})}\right) = \lim_{n\to\infty} \left(\frac{1}{1-\frac{1}{2^n}}\right) = \frac{1}{1} = 1.$$

Practice Problems. Evaluate the following limits.

11C.1. $\lim\limits_{x\to\infty} \dfrac{x^2+1}{4x^3-x+7}$

11C.2. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^3+1}}{x-3}$

11C.3. $\lim\limits_{x\to\infty} (\sqrt{x+1}-\sqrt{x-1})$

Answers

11C.4. $\lim\limits_{x\to-\infty} \dfrac{(x^3+1)^{1/3}}{x}$

11C.5. $\lim\limits_{x\to\infty} \sin x$

11C.6. $\lim\limits_{x\to\infty} (e^{-x}+e^x)$

Answers

D. Summary for Evaluating Limits

Consider the general expression $\lim\limits_{x\to a} f(x)$.

1. Remember that you know $x \to a$. The question is, does $f(x)$ get arbitrarily close to some value? If it does, then this value is the limit. (See Examples 11A.1, 11B.1, and Problem 11A.2.)

2. Many limits involve quotients of the form $\lim\limits_{x\to a} \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are continuous functions.

   If the quotient at $x = a$ has the form:

   a) $\dfrac{0}{\neq 0}$, $\Rightarrow$ limit is 0.
   (Examples 11B.2, 11B.5)

   b) $\dfrac{\neq 0}{0}$, $\Rightarrow$ $f(x)$ has a vertical asymptote at $x = a$, which means $f(x)$ becomes unbounded as $x \to a$. The answer is either ∞, $-\infty$, or does not exist. Evaluate f for values of x close to a and on either side of a to see which answer applies.
   (Examples 11B.3, 11B.6)

c) $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, $\Rightarrow$ you must do more manipulation to get an equivalent expression from which you can interpret the answer.
(Examples 11B.7, 11B.8, 11C.3, 11C.4)

Note: In Calc II you will learn L'Hospital's rule which is a great tool for analyzing limit problems of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

3. There are indeterminate forms such as $\infty - \infty$ where two operations work against each other or compete for control. (You will encounter more indeterminate forms in Calc II.) As above, you must manipulate the starting expression into an equivalent one that allows you to see what is happening.
(See Example 11C.2, Problem 11C.3)

Beginning of Topic Review Topics 250 Skills Assessment

B.1. a) $\displaystyle\lim_{x\to 2}\frac{2x^2-3x-2}{x^2-2}=\frac{8-6-2}{4-2}=\frac{0}{2}=0$ (Form $\frac{0}{\neq 0}$)

b) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{\ln(x+2)}=\frac{1-1}{\ln 2}=\frac{0}{\ln 2}=0$ (Form $\frac{0}{\neq 0}$)

Return to Review Topic

B.2. a) $\displaystyle\lim_{x\to 2^-}\frac{x}{x-2}$. This has the form $\frac{2}{0}$ $(\frac{\neq 0}{0})$, which means the function $\frac{x}{x-2}$ has a vertical asymptote at $x=2$ and thus becomes unbounded as x approaches 2 from either side. As $x\to 2^-$, $\frac{x}{x-2}$ is negative. Thus, $\displaystyle\lim_{x\to 2^-}\frac{x}{x-2}=-\infty$.

b) $\displaystyle\lim_{x\to\frac{\pi}{3}}\frac{\sin x}{2\cos x-1}$. This has the form

$$\frac{\sin\frac{\pi}{3}}{2\cos\frac{\pi}{3}-1}=\frac{\frac{\sqrt{3}}{2}}{2\cdot\frac{1}{2}-1}=\frac{\frac{\sqrt{3}}{2}}{0}\qquad(\text{form }\frac{\neq 0}{0}),$$

which means $x=\frac{\pi}{3}$ is a vertical asymptote. As $x\to\frac{\pi}{3}^-$, $(2\cos x-1)$ is positive (why?!) and $\sin x$ is positive. Thus $\displaystyle\lim_{x\to\frac{\pi}{3}^-}\frac{\sin x}{2\cos x-1}=\infty$. As $x\to\frac{\pi}{3}^+$, $(2\cos x-1)$ is negative and $\sin x$ is positive. This implies $\displaystyle\lim_{x\to\frac{\pi}{3}^+}\frac{\sin x}{2\cos x-1}=-\infty$. Since the limits from the left and right are not equal, $\displaystyle\lim_{x\to\frac{\pi}{3}}\frac{\sin x}{2\cos x-1}$ does not exist.

Return to Review Topic

11A.1. These answers come from studying the graph of $f(x)$ in Example 11A.2.

a) $\lim\limits_{x\to 0} f(x) = 1$

b) $\lim\limits_{x\to 3^+} f(x) = 1$

c) $\lim\limits_{x\to 3^-} f(x) = 3$

d) Based on b) and c), $\lim\limits_{x\to 3} f(x)$ does not exist.

Return to Problem

11A.2. a) Using 11A.4, $\lim\limits_{x\to 1} \dfrac{x+1}{x^2+2} = \dfrac{\lim\limits_{x\to 1} x+1}{\lim\limits_{x\to 1} x^2+2} = \dfrac{2}{3}$

b) Using 11A.4 and then 11A.1 on the denominator,

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin x}{x+\cos x} = \frac{\lim\limits_{x\to\pi/4} \sin x}{\lim\limits_{x\to\pi/4} (x+\cos x)}$$

$$= \frac{\lim\limits_{x\to\pi/4} \sin x}{\lim\limits_{x\to\pi/4} x + \lim\limits_{x\to\pi/4} \cos x} = \frac{\frac{1}{\sqrt{2}}}{\frac{\pi}{4} + \frac{1}{\sqrt{2}}}.$$

c) Using 11A.3, $\lim\limits_{x\to 2}(7-x)\ln x = \lim\limits_{x\to 2}(7-x)\lim\limits_{x\to 2}\ln x$

$$= 5\ln 2$$

Return to Problem

Find the following limits.

11B.1. $\lim_{x\to 1} \frac{\sin x}{\cos^2 x} = \frac{\sin(1)}{\cos^2(1)}$

Return to Problem

11B.2. $\lim_{x\to 1} \frac{x^2-1}{1-x}$ has form $\frac{0}{0}$. Try factoring.

$$\lim_{x\to 1} \frac{x^2-1}{1-x} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(1-x)} = \lim_{x\to 1} \frac{(x-1)(x+1)}{-(x-1)}$$
$$= \lim_{x\to 1} \frac{(x+1)}{-1} = \lim_{x\to 1} -(x+1) = -2$$

Return to Problem

11B.3. $\lim_{x\to 2} \frac{x+2}{x-2}$ has form $\frac{4}{0}$ $\left(\frac{\neq 0}{0}\right)$. This means the function $\frac{x+2}{x-2}$ has a vertical asymptote at $x = 2$. This further means that $\frac{x+2}{x-2}$ becomes unbounded as $x \longrightarrow 2^+$ and as $x \longrightarrow 2^-$. We must check both sides. As $x \longrightarrow 2^+$ $\frac{x+2}{x-2}$ is positive, and so $\lim_{x\to 2^+} \frac{x+2}{x-2} = \infty$. As $x \longrightarrow 2^-$ $\frac{x+2}{x-2}$ is negative, and so $\lim_{x\to 2^-} \frac{x+2}{x-2} = -\infty$. Thus, $\lim_{x\to 2} \frac{x+2}{x-2}$ does not exist.

Return to Problem

11B.4. $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$ has form $\frac{0}{0}$. Let's manipulate as follows.

$$\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{\frac{3-x}{3x}}{x-3} = \lim_{x\to 3} \frac{(3-x)}{(3x)(x-3)}$$
$$= \lim_{x\to 3} \frac{-1}{3x} = -\frac{1}{9}.$$

Return to Problem

11B.5. $\lim_{x\to 0} \dfrac{\cos x - 1}{x}$ has form $\dfrac{1-1}{0} = \dfrac{0}{0}$. This one is tricky. Use the hint and also remember Example 11B.7.

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x - 1}{x} &= \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \\
&= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{(-\sin x)}{\cos x + 1} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{(-\sin x)}{\cos x + 1} = 1 \cdot \frac{0}{2} = 1 \cdot 0 = 0.
\end{aligned}$$

NOTE: In Calc II you will learn L'Hospital's Rule which makes this problem easy.

Return to Problem

11B.6. Let $f(x) = \dfrac{1}{x}$, or $f(\ \ ) = \dfrac{1}{(\ \ )}$. Then,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim_{h\to 0} \frac{\dfrac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.
\end{aligned}$$

So, we have just proved the derivative of $\dfrac{1}{x}$ is $-\dfrac{1}{x^2}$.

Return to Problem

11C.1. $\lim_{x\to\infty} \dfrac{x^2+1}{4x^3-x+7}$. Since $x \to \infty$, let's factor x^2 from the numerator and x^3 from the denominator (the highest power possible in each case). Thus,

$$\lim_{x\to\infty} \frac{x^2+1}{4x^3-x+7} = \lim_{x\to\infty} \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^3\left(4-\frac{1}{x^2}+\frac{7}{x^3}\right)} = \lim_{x\to\infty} \frac{\left(1+\frac{1}{x^2}\right)}{x\left(4-\frac{1}{x^2}+\frac{7}{x^3}\right)}.$$

Note that $(1+\frac{1}{x^2}) \to 1$ and $(4-\frac{1}{x^2}+\frac{7}{x^3}) \to 4$ as $x \to \infty$. This means

$$\lim_{x\to\infty} \frac{\left(1+\frac{1}{x^2}\right)}{x\left(4-\frac{1}{x^2}+\frac{7}{x^3}\right)} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{\left(1+\frac{1}{x^2}\right)}{\left(4-\frac{1}{x^2}+\frac{7}{x^3}\right)} = 0.$$

NOTE: For the quotient of two polynomials, if $x \to \infty$ and the power of x in the denominator is greater than the power of x in the numerator, the limit is 0.

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11C.2. $\lim_{x\to\infty} \dfrac{\sqrt{x^3+1}}{x-3} = \lim_{x\to\infty} \dfrac{\sqrt{x^3\left(1+\frac{1}{x^3}\right)}}{x\left(1-\frac{3}{x}\right)} = \lim_{x\to\infty} \dfrac{x^{3/2}\sqrt{1+\frac{1}{x^3}}}{x\left(1-\frac{3}{x}\right)}$

$= \lim_{x\to\infty} \sqrt{x} \cdot \dfrac{\sqrt{1+\frac{1}{x^3}}}{\left(1-\frac{3}{x}\right)}$. Now $\sqrt{1+\frac{1}{x^3}} \to 1$ and $\left(1-\frac{3}{x}\right) \to 1$ as $x \to \infty$.

Thus,

$$\lim_{x\to\infty} \sqrt{x} \cdot \frac{\sqrt{1+\frac{1}{x^3}}}{\left(1-\frac{3}{x}\right)} = \infty.$$

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11C.3. $\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x-1})$ has the indeterminate form "$\infty-\infty$," which means that we must manipulate.

$$\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x-1}) = \lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x-1})\left(\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right)$$
$$= \lim_{x\to\infty}\frac{(x+1)-(x-1)}{\sqrt{x+1}+\sqrt{x-1}} = \lim_{x\to\infty}\left(\frac{2}{\sqrt{x+1}+\sqrt{x-1}}\right).$$

Since $\sqrt{x+1}$ and $\sqrt{x-1}$ both get arbitrarily large as $x\to\infty$,

$$\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x-1}) = \lim_{x\to\infty}\left(\frac{2}{\sqrt{x+1}+\sqrt{x-1}}\right) = 0.$$

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11C.4. $$\lim_{x\to-\infty}\frac{(x^3+1)^{1/3}}{x} = \lim_{x\to-\infty}\frac{\left(x^3\left(1+\frac{1}{x^3}\right)\right)^{1/3}}{x}$$
$$= \lim_{x\to-\infty}\frac{x\left(1+\frac{1}{x^3}\right)^{1/3}}{x} = \lim_{x\to-\infty}\left(1+\frac{1}{x^3}\right)^{1/3} = 1.$$

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11C.5. Find $\lim_{x\to\infty}\sin x$. No matter how large x becomes, $\sin x$ oscillates between -1 and 1. It does not "stay" arbitrarily close to any single value. Therefore, $\lim_{x\to\infty}\sin x$ does not exist.

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11C.6. As x gets large, $e^{-x}=\dfrac{1}{e^x}$ gets small since e^x gets large. Therefore $(e^{-x}+e^x)$ gets large as x gets large. This means $\lim_{x\to\infty}(e^{-x}+e^x)=\infty$.

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