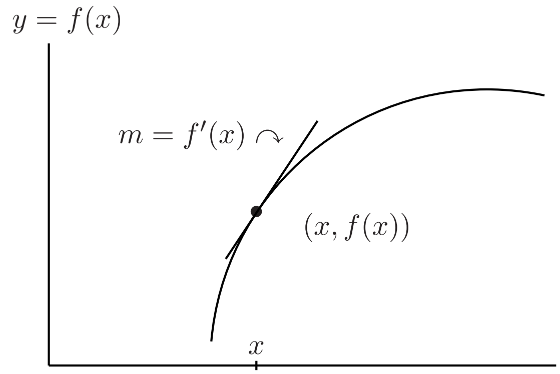
MATH 250 – TOPIC 12
DIFFERENTIATION: RULES AND APPLICATIONS

A. Definition and Common Functions

In this topic we review the formal definition of a derivative and list the derivatives of some common functions.

The derivative of a function at a particular value, x, represents the slope of the tangent line to the graph of the function at the point $(x, f(x))$.



The formal definition of the derivative is based on taking the limit of the difference quotient (see Review Topic 1).

$$\frac{df}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

For example, consider $f(x) = x^2$, (which really means $f(\ ) = (\ )^2$). Then,

$$\frac{df}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0}(2x + h) = 2x.$$

Here's another example of finding a derivative by using the definition. Suppose $f(x) = \sin x$.

$$\frac{df}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \sinh \cos x - \sin x}{h}$$

If we rearrange terms we see that the above limit is

$$\lim_{h\to 0} \left\{ \sin x \left\{ \frac{\cos h - 1}{h} \right\} + \cos x \left\{ \frac{\sin h}{h} \right\} \right\}.$$

Using the addition property of limits as well as the fact that $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ (Review Topic 12B) we have

$$f'(x) = \cos x.$$

Note: These two limits are evaluated in Example 11A.7 and Practice Problem 11B.5 in Review Topic 11.

Fortunately, we do not have to resort to the definition very often to calculate derivatives. The following table contains some common functions whose derivatives you should know by memory from Calc I. Complete the table and check your answers below. Memorize any derivatives you forgot.

Exercise A.1.

f	f'
C	
x^n	
e^x	
$\sin x$	
$\cos x$	
$\tan x$	
$\cot x$	
$\sec x$	
$\csc x$	
$\ln x$	

Answer

For the moment, let us concentrate on the power rule:

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

To utilize the power rule, a term must be written in the x^n form, where n is a number. This means that before differentiating, functions such as

$f = \sqrt[3]{x}$, $f = \dfrac{1}{x^2}$ and $f = \dfrac{1}{\sqrt{x}}$ must all be rewritten in the x^n form.

Examples:

$$\begin{aligned} f = x^3 &\quad\Rightarrow\quad f' = 3x^2 \\ f = \sqrt[3]{x} = x^{1/3} &\quad\Rightarrow\quad f' = \frac{1}{3}x^{-2/3} \\ f = \frac{1}{x^2} = x^{-2} &\quad\Rightarrow\quad f' = -2x^{-3} \\ f = \frac{1}{\sqrt{x}} = x^{-1/2} &\quad\Rightarrow\quad f' = -\frac{1}{2}x^{-3/2} \end{aligned}$$

Practice Problems. Find f' if

12A.1. $f(x) = \dfrac{1}{x^3}$ Answer

12A.2. $f(x) = \sqrt[4]{x}$ Answer

12A.3. $f(x) = \dfrac{1}{\sqrt{x^3}}$ Answer

B. Constant, Sum, and Difference Rules

There are a variety of rules we can utilize to find derivatives.

Let's start with the rule for finding the derivative of a basic constant multiplying a function, i.e.

$$\frac{d}{dx}(cf) = c\frac{df}{dx}.$$

The constant does not really come into play when you actually find the derivative. Some examples are:

$$\begin{aligned} f = 3x^2 &\quad\Rightarrow\quad f' = 3\frac{d}{dx}x^2 = 3(2x) = 6x \\ f = \pi e^x &\quad\Rightarrow\quad f' = \pi\frac{d}{dx}e^x = \pi(e^x) = \pi e^x \end{aligned}$$

Exercises. Find the following derivatives.

B.1. $f(x) = 4\sin x$ Answer

B.2. $f(x) = -3x^{3/2}$ Answer

B.3. $f(x) = -\dfrac{2}{x}$ Answer

Recall that the sum/difference rule is

$$\frac{d}{dx}(f \pm g) = f' \pm g'.$$

Here you can find the individual derivatives and then add or subtract the derivatives. Some examples are

$$\begin{aligned} g = x^3 + \tan x \quad &\Rightarrow \quad g' = 3x^2 + \sec^2 x \\ g = 4x^8 + x^{1/2} \quad &\Rightarrow \quad g' = 32x^7 + \frac{1}{2}x^{-1/2} \\ g = x^{1/3} - \csc x - \ln x \quad &\Rightarrow \quad g' = \frac{1}{3}x^{-2/3} + \csc x \cot x - \frac{1}{x} \end{aligned}$$

Practice Problems. Can you do the following problems?

12B.1. $g(x) = \sqrt[3]{x} + \ln x$ Answer

12B.2. $g(x) = \sin x + \tan x$ Answer

12B.3. $g(x) = \dfrac{1}{x^2} + e^x$ Answer

12B.4. $g(x) = x^2 - \dfrac{2}{x^3}$ Answer

12B.5. $g(x) = \dfrac{x^2 - x}{\sqrt{x}}$? Answer

C. Product Rule

We've stated some of the basic rules, but how do you differentiate something like $x^2 \sin x$ or $e^x \cos x$? The next rule gives us a formula to differentiate products of factors. (You can derive this rule using the definition of derivative and basic limit rules.)

$$\frac{d}{dx}(fg) = f'g + fg'$$

To remind yourself of this rule you may want to say, "the derivative of the first times the second, plus the first times the derivative of the second." Some examples are

$$\begin{aligned}
f &= (x^2+2)\cot x \\
&\Rightarrow f' = \left[\frac{d}{dx}(x^2+2)\right]\cot x + (x^2+2)\frac{d}{dx}\cot x && \text{Set-up step} \\
&= 2x\cot x + (x^2+2)(-\csc^2 x) && \text{Taking derivatives}
\end{aligned}$$

$$\begin{aligned}
f &= (\sin x \sec x) \\
&\Rightarrow f' = \left(\frac{d}{dx}\sin x\right)\sec x + \sin x\left(\frac{d}{dx}\sec x\right) \\
&\qquad\qquad (\cos x)\sec x + \sin x(\sec x \tan x) \\
&= 1 + \tan^2 x = \sec^2 x
\end{aligned}$$

$$\begin{aligned}
f &= x \ln x \\
&\Rightarrow f' = 1 \ln x + x\left(\frac{1}{x}\right) = \ln x + 1
\end{aligned}$$

Comment: Notation like $\frac{d}{dx}$, $\frac{d}{dt}$, etc. simply announces your intent to differentiate a term or expression. With a more complicated function, a "set-up" step may be necessary before you actually take derivatives.

Practice Problems. Can you find f' if

12C.1. $f(x) = (x^2 + 3)e^x$ Answer

12C.2. $f(x) = (\sin x)\ln x$ Answer

12C.3. $f(x) = (x^2 + 3x - 1)(\ln x)\tan x$? Answer

D. Quotient Rule

The quotient rule is used to calculate the derivative of functions such as

$$\frac{x^2 + \ln x}{5 + \cos x}.$$

This function involves a quotient of two other functions. i.e. $\left[\dfrac{f(x)}{g(x)}\right]$. The derivative of such a function is given by

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'g - fg'}{g^2}.$$

How would you say this rule? "The derivative of the numerator times the denominator minus the numerator times the derivative of the denominator all divided by the denominator squared." Here are some examples.

$$f = \frac{x^3+3}{x^2-1} \quad \Rightarrow \quad f' = \frac{\left[\dfrac{d}{dx}(x^3+3)\right](x^2-1) - (x^3+3)\dfrac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{3x^2(x^2-1) - (x^3+3)2x}{(x^2-1)^2}$$

$$f = \frac{e^x+4}{1+\ln x} \quad \Rightarrow \quad f' = \frac{\left[\dfrac{d}{dx}(e^x+4)\right](1+\ln x) - (e^x+4)\dfrac{d}{dx}(1+\ln x)}{(1+\ln x)^2}$$

$$\times \frac{e^x(1+\ln x) - (e^x+4)\dfrac{1}{x}}{(1+\ln x)^2}$$

$$f = \frac{x^2 - \sin x}{x^{1/4} + x} \quad \Rightarrow \quad f' = \frac{(2x - \cos x)(x^{1/4} - x) - (x^2 - \sin x)\left(\frac{1}{4}x^{-3/4} + 1\right)}{(x^{1/4} + x)^2}$$

Practice Problems. Can you differentiate the following?

12D.1. $f(x) = \dfrac{x + e^x}{\sin x + \cos x}$ Answer

12D.2. $f(x) = \dfrac{1 + \ln x}{1 - \ln x}$ Answer

12D.3. $f(x) = \dfrac{x^2 - 2x + 3}{3x^4 - x + 2}$ Answer

E. Chain Rule

The chain rule deals with differentiating composite functions. It enables us to find the derivatives of complicated expressions not covered by the rules in the previous review topics. To apply the chain rule we must first understand composite functions.

The term composite function describes the situation when one function is plugged into another. The notation $f(g(x))$ represents a composite function. As an example, let's begin with the function $f(x) = \sin x$. This really means

$$f(\;) = \sin(\;), \text{ (use a blank for the input or argument)}$$

Now put another function $g(x)$ as the input for $f(\;)$. That is, consider $f(g(x))$, which means $g(x)$ is plugged into $f(\;)$. Thus we have $f(g(x)) = \sin(g(x))$. If $g(x) = x^2+1$, then $f(g(x)) = \sin(g(x)) = \sin(x^2+1)$.

Roughly speaking, you can think of $f(\;) = \sin(\;)$ as the "outside" function and $g(x) = x^2 + 1$ as the function "inside" f.

As another example, let $f(x) = x^3$ and $g(x) = \cos x$. Find $f(g(x))$. The notation $f(g(x))$ means that $g(x)$ is plugged into f, and so $f(\ ) = (\ )^3$ is the outside function (use the blanks!) and $g(x) = \cos x$ is the inside function. Thus

$$f(g(x)) = (g(x))^3 = (\cos x)^3$$

Exercise E.1. If $f(x) = x^3$ and $g(x) = \cos x$, find $g(f(x))$ and $f(f(x))$.

Answer

Given $f(x)$ and $g(x)$, computationally it is easy to determine $f(g(x))$ or $g(f(x))$. Consider instead the function $\ln(x^2+2)$. Can you find an $f(x)$ and $g(x)$ so that $f(g(x)) = \ln(x^2+2)$?

One answer would be $g(x) = x^2$ and $f(x) = \ln(x+2)$. Can you find another?

Composites are not limited to two functions.

Example: Let $f(x) = e^x$, $g(x) = \sqrt{x}$, and $h(x) = x^2 + 2$. Find $f(g(h(x))$.

Well, $f(\ ) = e^{(\ )}$ is the outside function, $g(\ ) = \sqrt{(\ )}$ is the middle function and is plugged into f, and finally $h(x) = x^2 + 2$ is the inside function and is plugged into g. Thus,

$$f(g(h(x)) = e^{(g(h(x)))} = e^{\sqrt{h(x)}} = e^{\sqrt{x^2+1}}.$$

Exercise E.2. Find $h(f(g(x)))$. Answer

There is another way to determine outside and inside functions in a composition. Let's go back to $f(x) = \sin(x^2+1)$. If you wanted to compute $f(2)$ you would first need to compute 2^2+1 and then find the $\sin(5)$. This would mean $\sin(\ )$ is the "outside" function and x^2+1 is the "inside" function.

Differentiation

To differentiate composite functions we use the **chain rule**, which is written as follows:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = f'(\ ) \cdot g'(x)$$

In words, this means differentiate the <u>outside</u> function, leave the argument of the outside function alone, and then <u>multiply</u> by the derivative of the inside function.

Example: Differentiate $\sin(x^2 + 1)$.

As seen earlier, this is a composite function with outside function sin( ) and inside function $x^2 + 1$. Thus,

$$\begin{aligned}\frac{d}{dx}[\sin(x^2+1)] &= [\sin(\ )]' \cdot \frac{d}{dx}(x^2+1) \\ &= \cos(\ ) \cdot (2x) \\ &= \cos(x^2+1)(2x)\end{aligned}$$

Notice that the word definition of the chain rule matches the above calculation exactly. We differentiated the outside function, left the <u>argument</u> of the outside function alone, and then multiplied by the derivative of the inside function.

Example: Differentiate $\sqrt{x^3+x}$.

First, rewrite $\sqrt{x^3+x}$ as $(x^3+x)^{1/2}$. (Recall that it is usually better to replace radical signs by exponents.) Then $(x^3+x)^{1/2}$ is a composite function with outside function $(\ )^{1/2}$ and inside function (x^3+x). By the chain rule,

$$\begin{aligned}
\frac{d}{dx}\left[(x^3+x)^{1/2}\right] &= \left[(\quad)^{1/2}\right]' \cdot \frac{d}{dx}[x^3+x] \\
&= \frac{1}{2}(\quad)^{-1/2} \cdot (3x^2+1) \\
&= \frac{1}{2}(x^3+x)^{-1/2} \cdot (3x^2+1) \\
&= \frac{3x^2+1}{2\sqrt{x^3+x}}
\end{aligned}$$

Some additional examples are given below.

$$\begin{aligned}
f = \ln(x^2+2);\, f' &= [\ln(\ )]' \cdot \frac{d}{dx}[x^2+1] \\
&= \frac{1}{(\ )} \cdot 2x = \frac{2x}{x^2+1}
\end{aligned}$$

$$\begin{aligned}
f = \sec(\ln x);\, f' &= [\sec(\ )]' \cdot \frac{d}{dx}[\ln x] \\
&= \sec(\ )\tan(\ ) \cdot \frac{1}{x} \\
&= \sec(\ln x)\tan(\ln x) \cdot \frac{1}{x}
\end{aligned}$$

As you become proficient, you can omit some of the middle steps.

$$f = e^{\sin x}; \quad f' = e^{\sin x} \cdot \cos x$$

As a final "challenge," let's differentiate one of our earlier examples, $f(x) = e^{\sqrt{x^2+1}}$. There are 3 functions in this composition: $e^{\{\ \}}$, $(\ )^{1/2}$, and (x^2+1). To find the derivative, work from the outside in and keep multiplying by the derivative of the "function inside."

$$\begin{aligned} f' &= \left[e^{\{\ \}}\right]' \cdot \frac{d}{dx}[\{\ \}] \\ &= e^{\{\ \}} \cdot \frac{d}{dx}(x^2+1)^{1/2} \\ &= e^{(x^2+1)^{1/2}} \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x) \\ &= \frac{x\, e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}} \end{aligned}$$

Practice Problems. Differentiate the following.

12E.1. $y = \ln\cos x$ Answer

12E.2. $y = (x^2+1)^{1/2}$ Answer

12E.3. $f(x) = \sec(e^x+1)$ Answer

12E.4. $f(x) = (1+\sin(x^2+3))^{2/3}$ Answer

F. Implicit Differentiation

Before we introduce the next method a word should be said about dummy variables. If $y = x^2$, then y' means $\frac{dy}{dx}$ and you arrive at $y' = 2x$. If $y = t^2$ then y' means $\frac{dy}{dt}$ and you arrive at $y' = 2t$.

You could just as easily let $y =(*)^2$. Then y' means $\frac{dy}{d(*)}$ and you arrive at $y' = 2(*)$. The variables x, t, and $*$ are "dummy" variables. It is understood that in each case they represent the independent variable and y is the dependent variable. If you are asked to find $\frac{dy}{dz}$, then z is understood to be the independent variable and y depends upon z.

Do not let the notation catch you. You also need to leave the mindset that x is always the independent variable and y the dependent variable. You could just as easily have let y be the independent variable and x the dependent variable leading to $\dfrac{dx}{dy}$.

An implicit function is a function where you cannot write y explicitly as a function of x or x as a function of y.

Normally functions have the form $y = f(x)$. For example,

$$y = x^3 + x^2. \tag{12.1}$$

The dependent variable or output y is all by itself on one side of the equation while the x terms are on the other side. In a situation like this we say y is <u>explicitly</u> defined in terms of x. Also, you can easily make a table where x is the input and y is the output.

x	0	1	-1	2
y	0	2	0	12

etc.

An <u>implicit</u> function has a different appearance. Consider the equation

$$x + y = 1 + e^{xy}. \tag{12.2}$$

If $x = 0$, then what value must y be in order to make equation (12.2) true? Answer: $y = 2$. That is, $(x, y) = (0, 2)$ satisfies (12.2). If $x = 2$, then $y = 0$; (i.e., $(2, 0)$ also makes equation (12.2) true). So, given x, theoretically there is a y such that the pair (x, y) satisfies equation (12.2). Thus in both equations (12.1) and (12.2) we can consider x as an input that gets mapped to an output y, which is exactly the definition of a function. The mixing of x and y in (12.2) prevents us from solving for y in terms of x, so we say y is <u>implicitly</u> defined in terms of x.

As another example, consider

$$y^3 - x^3 = x - y. \tag{12.3}$$

Since we have a rule which takes inputs x to outputs y, equation (12.3) defines a function y implicitly in terms of x.

Next, how do we differentiate implicit functions? Easy. Consider equation (12.2). Remember that y depends on x; *you just don't know how.*

Now differentiate both sides of (12.2) with respect to x. We have

$$\frac{d}{dx}[x + y(x)] = \frac{d}{dx}[e^{xy(x)}]. \tag{12.4}$$

Note: On the left-hand side we use the sum rule. On the right side we must use the chain rule (see previous Review Topic 12E) and the product rule. Equation (12.4) leads to:

$$1 + \frac{dy}{dx} = \frac{d}{dx}[e^{xy(x)}] = e^{xy(x)}\frac{d}{dx}[xy(x)] = e^{xy(x)}\left(y + x\frac{dy}{dx}\right),$$

or

$$1 + y' = e^{xy}(y + xy').$$

Now solve for y'.

$$\begin{aligned} y' - e^{xy}xy' &= e^{xy}y - 1 \\ y' &= \frac{e^{xy}y - 1}{1 - xe^{xy}}. \end{aligned}$$

Exercise F.1. Actually evaluate $y'(0)$ and explain what it represents.

Answer

Method for Implicit Differentiation

A. Suppose y is defined implicitly in terms of x in some type of equation like (12.2) or (12.3).

B. Formally differentiate both sides of the equation remembering that $y = y(x)$ is a function of x.

C. Solve for y'. (You can <u>always</u> solve for y' if you did Step B correctly).

Let's apply the method to find y' where y is defined implicitly by equation (12.3).

$$\begin{aligned} y^3 - x^3 &= x - y \\ \frac{d}{dx}(y(x)^3 - x^3) &= \frac{d}{dx}(x - y(x)). \end{aligned}$$

Note: Differentiating y^3 requires the chain rule.

$$\begin{aligned} 3y^2y' - 3x^2 &= 1 - y' \\ y'(1+3y^2) &= 1+3x^2 \\ y' &= \frac{1+3x^2}{1+3y^2} \end{aligned}$$

As a final example, find y' if $2xy + y^2 = x^3$.

$$\frac{d}{dx}[2xy(x) + (y(x))^2] = \frac{d}{dx}[x^3].$$

Note: $2xy = (2x)(y(x))$ and so we must use the product rule.

$$\begin{aligned} 2y + 2xy' + 2yy' &= 3x^2 \\ y'(2x+2y) &= 3x^2 - 2y \\ y' &= \frac{3x^2-2y}{2x+2y} \end{aligned}$$

Practice Problems. Find y' if

12F.1. $y = \ln(x+y)$ Answer

12F.2. $x^2y + 3xy^3 - 1 = 0$ Answer

12F.3. $\tan y = x$ Answer

12F.4. $x + y = e^{x+y^2}$ Answer

G. Logarithmic Differentiation (Includes $\mathbf{y = x^x}$)

In order to understand logarithmic differentiation, you must feel comfortable with the Chain Rule Review Topic 12E and Implicit Differentiation Review Topic 12F.

We know how to find the derivative when $y = e^{f(x)}$ (constant base and variable in the exponent) or $y = x^m$ (variable in the base and constant exponent). What do we do when $y = (f(x))^{g(x)}$ (variables in both the exponent and base)? The key is to take the natural logarithm of both sides, i.e., $\ln y = \ln(f)^g = g \ln f(x)$. Now we use implicit differentiation to compute the derivative.

Examples:

1) Find y' if $y = x^{(x^2+1)}$. Note: Variables in both the base and exponent. Taking the natural log of both sides we see

$$\ln y = \ln x^{(x^2+1)} = (x^2+1)\ln x$$

Differentiating both sides implicitly:

$$\frac{1}{y}y' = 2x\ln x + (x^2+1)\frac{1}{x}$$

Solving for $y' \Rightarrow y' = y\left[2x\ln x + \dfrac{x^2+1}{x}\right] = x^{x^2+1}\left[2x\ln x + \dfrac{x^2+1}{x}\right]$.

2)
$$\begin{aligned} y &= (x^2+1)^{\sin x} \\ \ln y &= \ln(x^2+1)^{\sin x} = \sin x \ln(x^2+1) \\ &\Rightarrow \frac{1}{y}y' = \cos x \ln(x^2+1) + \sin x \frac{2x}{x^2+1} \\ &\Rightarrow y' = (x^2+1)^{\sin x}\left[\cos x \ln(x^2+1) + \frac{2x\sin x}{x^2+1}\right] \end{aligned}$$

Remark: Logarithmic differentation also works for functions like 3^{x^2+1} (a constant base and variable exponent), in case you forget the formula for functions like this.

Practice Problems. Find y' if

12G.1. $y = (x^3 + 3x - 1)^{x^2}$ Answer

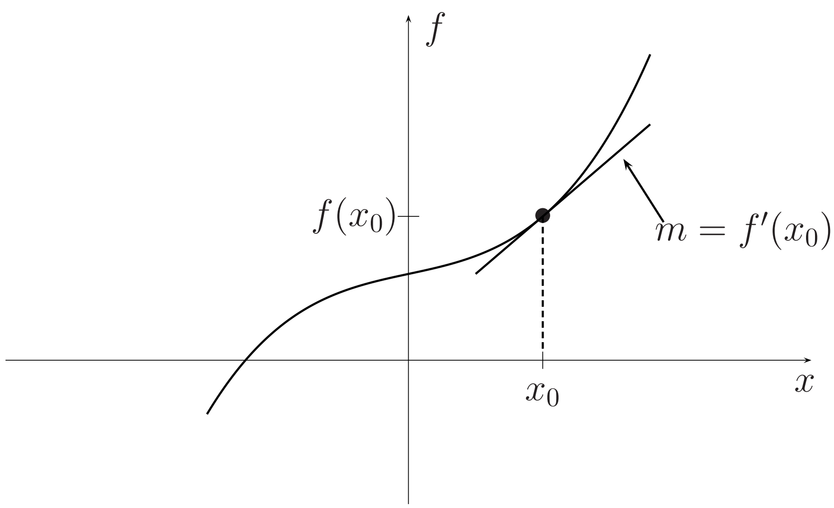
12G.2. $y = (2 + \cos x)^x$ Answer

12G.3. $y = (\ln x)^{x^2+2x+4}$ Answer

12G.4. $y = 2^{\sin x}$ Answer

H. Tangent Lines

Remember, the derivative is the slope of the tangent line to the curve at a particular point. Thus $f'(x_0)$ is the slope of the tangent line at the point $(x_0, f(x_0))$.



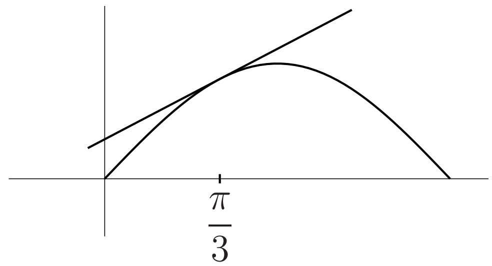
Example: Consider $f(x) = \sin x$. Find the point on the curve and the slope of the tangent line at $x_0 = \dfrac{\pi}{3}$.

$$\text{At } x_0 = \frac{\pi}{3}, \quad f(x_0) = f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\text{and the point is } \left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

To find the slope we compute $f'(x) = \cos x$.

$$\text{At } x_0 = \frac{\pi}{3}, \quad m = f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}. \text{ The slope is } \frac{1}{2}.$$



One of the first applications of differentiation in Calculus I was finding the equation of a tangent line. The basic equation of a line through the point (x_0, y_0) is

$$y - y_0 = m(x - x_0),$$

where m is the slope and (x_0, y_0) is a point on the line. In calculus, the slope is the derivative of the curve at the point (x_0, y_0) i.e., $m = f'(x_0)$. If you keep the idea that points on the curve come from f and slopes of tangent lines come from f', then finding the equation of the tangent line is not very difficult.

Examples:

1) Find the equation of the tangent line to the curve $f(x) = e^{2x+1} + x$ at $(0, e)$.

The point is given: $(0, e)$

To find the slope we need the derivative which is given by $f'(x) = 2e^{2x+1} + 1$. Evaluating this at the given point we find $m = f'(0) = 2e + 1$. Thus

$$y - e = (2e + 1)(x - 0)$$

is the equation of the tangent line to the curve $f(x) = e^{2x+1} + x$ at $(0, e)$.

2) Find the equation of the tangent line to the curve $f(x) = \ln(x^2 - 2x + 1)$ where the curve crosses the positive x-axis.

This problem is a little more challenging. To find the point on the line, note that the curve crosses the x-axis when $y = 0$. So we need $\ln(x^2 - 2x + 1) = 0$ or $x^2 - 2x + 1 = 1$. Solving for x we have $x(x - 2) = 0$. The only positive x-value is $x = 2$, and so the point is $(2, 0)$.

To find the slope, we compute the derivative

$$f' = \frac{2x + 2}{x^2 - 2x + 1} \Rightarrow f'(2) = \frac{6}{1} = 6 = m.$$

Thus

$$y - 0 = 6(x - 2)$$

is the equation of the tangent line to the curve $f(x) = \ln(x^2 - 2x + 1)$ when the curve crosses the positive x-axis.

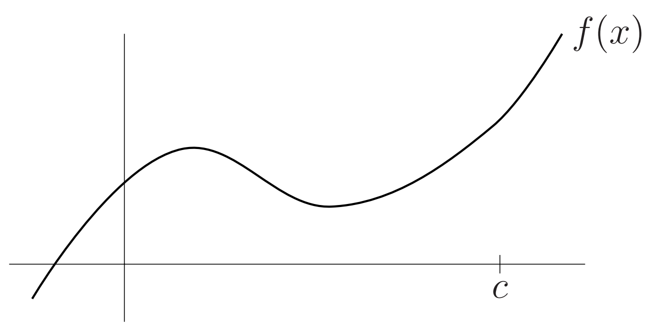
Now let's see if you really understand the concept of derivative and tangent line.

Exercises.

H.1. Given the graph of f, describe what is being measured by the following:

a) $f(c)$
b) $f'(c)$
c) $f(0)$
d) Solving $f(x) = 0$ for x
e) Solving $f'(x) = 0$ for x

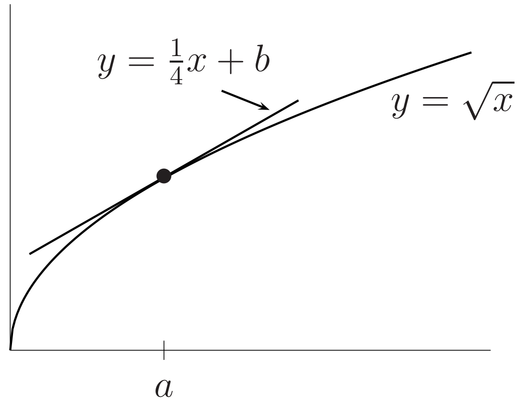
Answers



Now we will do 3 examples of these concepts.

H.2. Choose any function you like and any x value in its domain. Find the equation of the associated tragent line. Answer

H.3.



The line $y = \dfrac{1}{4}x + b$ is tangent to the graph $f(x) = \sqrt{x}$. Find values of a and b. Answer

H.4. Given $y^2 = xy - 4x$, find all points where tangent lines to the curve are a) horizontal or b) vertical. Answer

Practice Problems.

12H.1. Find the equation of the tangent line for $f(x) = x^3 + x - 4$ at $(1, -2)$. Answer

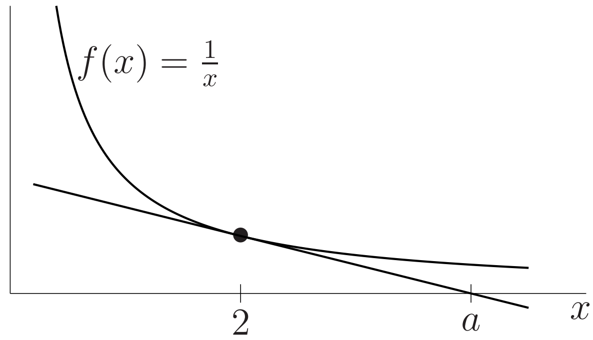
12H.2. Find the equation of the tangent line for $g(x) = \ln(x^2 + 2x + 4)$ when $x = 2$. Answer

12H.3. Find the equation of the tangent line for $y+x = 1+\cos(xy)$ when the curve crosses the x-axis. Answer

12H.4. Given $f(x) = x^3 - 2x^2 + 5$, find the points on the curve where the tangent line is horizontal. Answer

12H.5. Given $y^2 = 2x - 6$, find the points on the curve where the tangent line is vertical. Answer

12H.6. The straight line is tangent to the graph of $f(x) = \dfrac{1}{x}$. Find the value of a. Answer



Beginning of Topic Review Topics 250 Skills Assessment

A.1.

f	f'
C	0
x^n	$n\,x^{n-1}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$\csc x \cot x$
$\ln x$	$\dfrac{1}{x}$

As in our example for $\sin x$, we could have derived each of the rules by going back to the definition of derivative and using properties of limits. Maybe you are just as happy we didn't?!

Return to Review Topic

12A.l. $f(x) = \dfrac{1}{x^3}$

12A.2. $f(x) = \sqrt[4]{x}$

12A.3. $f(x) = \dfrac{1}{\sqrt{x^3}}$

Answers:

12A.1. $f = x^{-3} \Rightarrow f' = -\dfrac{3}{x^4}$

12A.2. $f = x^{1/4} \Rightarrow f' = \dfrac{1}{4}x^{-3/4} \Rightarrow f' = \dfrac{1}{4\sqrt[4]{x^3}}$

12A.3. $f = x^{-3/2} \Rightarrow f' = -\dfrac{3}{2}x^{-5/2} \Rightarrow f' = -\dfrac{3}{2\sqrt{x^5}}$

Return to Problem

B.1. $f(x) = 4\sin x$

B.2. $f(x) = -3x^{3/2}$

B.3. $f(x) = -\dfrac{2}{x}$

Answers:

B.1. $f' = 4\cos x$

B.2. $f' = -3(\dfrac{3}{2})x^{1/2} = -\dfrac{9}{2}x^{1/2}$

B.3. $f' = \dfrac{2}{x^2}$

Return to Review Topic

12B.1. $g(x) = \sqrt[3]{x} + \ln x$

12B.2. $g(x) = \sin x + \tan x$

12B.3. $g(x) = \dfrac{1}{x^2} + e^x$

12B.4. $g(x) = x^2 - \dfrac{2}{x^3}$

12B.5. $g(x) = \dfrac{x^2 - x}{\sqrt{x}}$?

Answers:

12B.1. $g'(x) = \dfrac{1}{3}x^{-2/3} + \dfrac{1}{x}$

12B.2. $g' = \cos x + \sec^2 x$

12B.3. $g = x^{-2} + e^x \Rightarrow g' = -2x^{-3} + e^x \Rightarrow g' = -\dfrac{2}{x^3} + e^x$

12B.4. $g = x^2 - 2x^{-3} \Rightarrow g' = 2x - 2(-3)x^{-4} = 2x + \dfrac{6}{x^4}$

12B.5. $g = x^{3/2} - x^{1/2} \Rightarrow g' = \dfrac{3}{2}x^{1/2} - \dfrac{1}{2}x^{-1/2}$

Return to Problem

12C.1. $f(x) = (x^2 + 3)e^x$

12C.2. $f(x) = (\sin x) \ln x$

12C.3. $f(x) = (x^2 + 3x - 1)(\ln x) \tan x$?

Answers:

12C.1. $g' = 2xe^x + (x^2 + 3)e^x$

12C.2. $f' = \cos x \ln x + (\sin x)\dfrac{1}{x}$

12C.3.
$$\begin{aligned} f' &= \Big[\frac{d}{dx}(x^2 + 3x - 1)\Big](\ln x) \tan x + (x^2 + 3x - 1)\frac{d}{dx}((\ln x) \tan x) \\ &= (2x + 3) \ln x \tan x + (x^2 + 3x - 1)\Big[\frac{1}{x} \tan x + (\ln x) \sec^2 x\Big] \end{aligned}$$

Return to Problem

12D.1. $f(x) = \dfrac{x + e^x}{\sin x + \cos x}$

12D.2. $f(x) = \dfrac{1 + \ln x}{1 - \ln x}$

12D.3. $f(x) = \dfrac{x^2 - 2x + 3}{3x^4 - x + 2}$

Answers:

12D.1.
$$f' = \frac{\big(\frac{d}{dx}(x + e^x)\big)(\sin x + \cos x) - (x + e^x)\frac{d}{dx}(\sin x + \cos x)}{(\sin x + \cos x)^2}$$
$$= \frac{(1 + e^x)(\sin x + \cos x) + (1 + e^x)(\sin x - \cos x)}{(\sin x + \cos x)^2}$$

12D.2.
$$f' = \frac{\big[\frac{d}{dx}(1 - \ln x)\big](1 - \ln x) - (1 + \ln x)\frac{d}{dx}(1 - \ln x)}{(1 - \ln x)^2}$$
$$= \frac{\frac{1}{x}(1 - \ln x) - (1 + \ln x)\big(-\frac{1}{x}\big)}{(1 - \ln x)^2}$$

12D.3.
$$f' = \frac{(2x - 2)(3x^4 - x + 2) - (x^2 - 2x + 3)(12x^3 - 1)}{(3x^4 - x + 2)^2}$$

Return to Problem

E.1. If $f(x) = x^3$ and $g(x) = \cos x$, find $g(f(x))$ and $f(f(x))$.

E.2. Find $h(f(g(x)))$.

Answers:

E.1.
$$g(f(x)) = g(x^3) = \cos(x^3)$$
$$f(f(x)) = f(x^3) = (x^3)^3 = x^9$$

E.2.
$$\begin{aligned} h(f(g(x))) &= h(f(\sqrt{x})) = h(e^{\sqrt{x}}) \\ &= (e^{\sqrt{x}})^2 = e^{2\sqrt{x}} + 2 \end{aligned}$$

Return to Review Topic

12E.1. $y' = \dfrac{1}{\cos x} \cdot \dfrac{d}{dx}(\cos x) = \dfrac{1}{\cos x}(-\sin x) = -\tan x$

Return to Problem

12E.2. $y' = \dfrac{1}{2}(x^2+1)^{-1/2}\dfrac{d}{dx}(x^2+1) = \dfrac{x}{\sqrt{x^2+1}}$

Return to Problem

12E.3.
$$\begin{aligned} f' &= \sec(e^x+1)\tan(e^x+1)\frac{d}{dx}(e^x+1) \\ &= \sec(e^x+1)\tan(e^x+1)e^x \end{aligned}$$

Return to Problem

12E.4.
$$\begin{aligned} f' &= \frac{2}{3}(1+\sin(x^2+3))^{-1/3}\frac{d}{dx}(1+\sin(x^2+3)) \\ &= \frac{2}{3}(1+\sin(x^2+3))^{-1/3}(\cos(x^2+2))\frac{d}{dx}(x^2+2) \\ &= \frac{2}{3}(1+\sin(x^2+3))^{-1/3}(\cos(x^2+2)) \cdot 2x \end{aligned}$$

Return to Problem

F.1. Evaluate $y'(0)$.

Answer:

F.1. When $x = 0$, $y = 2$

$$y'(0) = \frac{e^0 \cdot 2 - 1}{1 - 0 \cdot e^0} = 1.$$

$y'(0)$ indicates the tangent line to the curve at the point $(0, 2)$ has slope 1.

Return to Review Topic

12F.1. $$y' = \frac{1}{x+y}(1+y')$$
$$y' = \frac{1}{x+y-1}$$

Return to Problem

12F.2. $$2xy + x^2y' + 3y^3 + (3x)3y^2y' = 0$$
$$y' = \frac{-2xy - 3y^3}{x^2 + 9xy^2}$$

Return to Problem

12F.3. $$\sec^2 y\, y' = 1$$
$$y' = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1}$$

Notice if we rewrote this equation as $y = \tan^{-1} x$ then $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{x^2+1}$.

Return to Problem

12F.4. $$1 + y' = e^{x+y^2}[1 + 2yy']$$
$$y' = \frac{1 - e^{x+y^2}}{2ye^{x+y^2} - 1}$$

Return to Problem

12G.1.
$$\ln y = x^2 \ln(x^3 + 3x - 1)$$
$$\frac{1}{y} y' = 2x \ln(x^3 + 3x - 1) + x^2 \frac{1}{x^3 + 3x - 1}(3x^2 + 3)$$
$$y' = (x^3 + 3x - 1)^{x^2} \left[2x \ln(x^3 + 3x - 1) + \frac{x^2(3x^2 + 3)}{x^3 + 3x - 1} \right]$$

Return to Problem

12G.2.
$$\ln y = x \ln(2 + \cos x)$$
$$\frac{1}{y} y' = \ln(2 + \cos x) + x \frac{1}{2 + \cos x}(-\sin x)$$
$$y' = (2 + \cos x)^x \left[\ln(2 + \cos x) - \frac{x \sin x}{2 + \cos x} \right]$$

Return to Problem

12G.3.
$$\ln y = (x^2 + 2x + 4) \ln(\ln x)$$
$$\frac{1}{y} y' = (2x + 2) \ln(\ln x) + (x^2 + 2x + 4) \frac{1}{\ln x} \cdot \frac{1}{x}$$
$$y' = (\ln x)^{x^2 + 2x + 4} \left[(2x + 2) \ln(\ln x) + \frac{x^2 + 2x + 4}{x \ln x} \right]$$

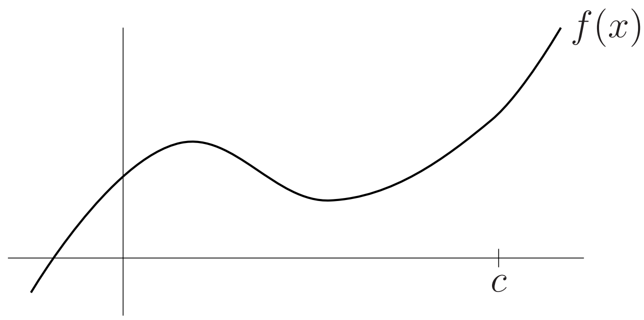
Return to Problem

12G.4.
$$\ln y = \sin x \ln 2$$
$$\frac{1}{y} y' = \cos x \ln 2 + \sin x \cdot 0$$
$$y' = 2^{\sin x} (\cos x) \ln 2$$

Return to Problem

H.1. Given the graph of f, describe what is being measured by the following:

a) $f(c)$
b) $f'(c)$
c) $f(0)$
d) Solving $f(x) = 0$ for x
e) Solving $f'(x) = 0$ for x



Answers:

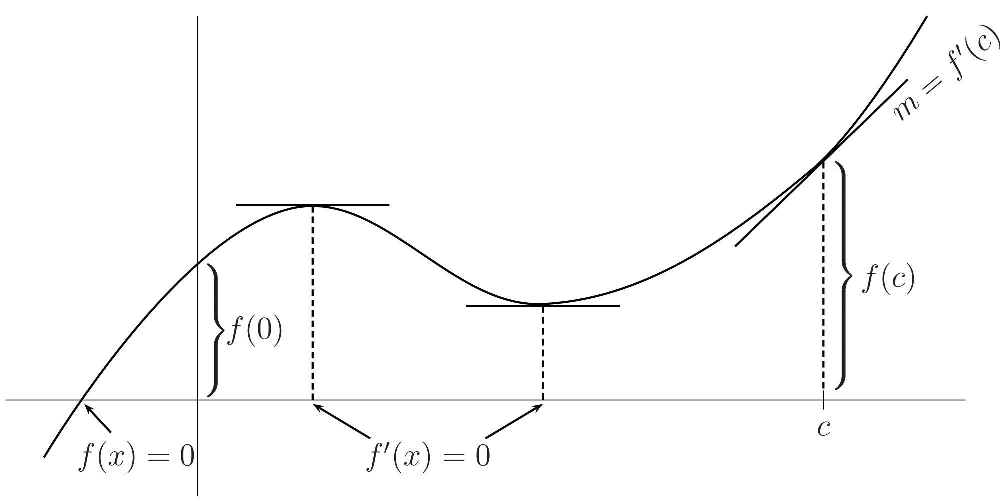
a) $f(c)$ is the output of the function when c is the input.

b) $f'(c)$ is the slope of the tangent line at $x = c$.

c) $f(0)$ is the y-intercept (output of f when 0 is input).

d) Solving $f(x) = 0$ find the zeroes (x-intercepts) of f.

e) Solving $f'(x) = 0$ finds all x where the tangent line to f is horizontal.



Return to Review Topic

H.2. Any tangent line problem you prefer.

Answer:

H.2. Consider $x = 1$.

$f(1) = \sqrt{1} = 1$, $f' = \dfrac{1}{2}x^{-1/2}$, $f'(1) = \dfrac{1}{2}$. The point is $(1, 1)$ and the slope is $\dfrac{1}{2}$. The equation of the tangent line is $y - 1 = \dfrac{1}{2}(x - 1)$.

Consider $x = 9$.

$F(9) = \sqrt{9} = 3$, $f'(9) = \dfrac{1}{2}9^{-1/2} = \dfrac{1}{6}$. The point is $(9, 3)$ and the slope is $\dfrac{1}{6}$. The equation of the tangent line is

$$y - 3 = \frac{1}{6}(x - 9).$$

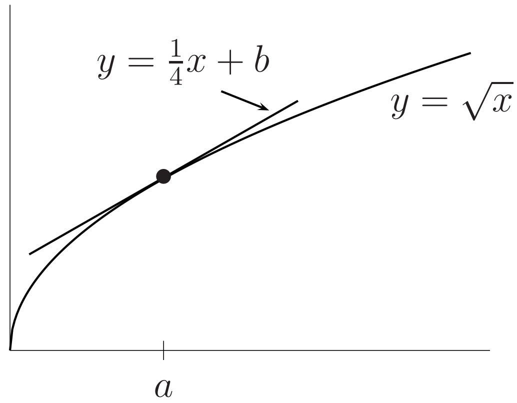
Consider $x = 7$.

$f(7) = \sqrt{7}$, $f'(7) = \dfrac{1}{2}7^{-1/2} = \dfrac{1}{2\sqrt{7}}$. The point is $(7, \sqrt{7})$ and the slope is $\dfrac{1}{2\sqrt{7}}$. The equation of the tangent line is

$$y - \sqrt{7} = \frac{1}{2\sqrt{7}}(x - 7).$$

Return to Review Topic

H.3.



The line $y = \dfrac{1}{4}x + b$ is tangent to the graph $f(x) = \sqrt{x}$. Find values of a and b.

Answer:

H.3. The slope of the tangent line is $\dfrac{1}{4}$. Set $f' = \dfrac{1}{4} = \dfrac{1}{2x^{1/2}} \Rightarrow x^{1/2} = 2 \Rightarrow x = 4$. $F(4) = 2 = a$. The equation of the tangent line is

$$y - 2 = \frac{1}{4}(x - 4) \Rightarrow y = \frac{1}{4} + 3.$$

Return to Review Topic

H.4. Given $y^2 = xy - 4x$, find all points where tangent lines to the curve are a) horizontal or b) vertical.

Answer:

H.4. Using implicit differentiation we have

$$2yy' = y + xy' - 4.$$

Solving for y'

$$y' = \frac{y-4}{2y-x}.$$

Horizontal tangents occur when $y - 4 = 0 \Rightarrow y = 4$.
Vertical tangents occur when $2y - x = 0 \Rightarrow y = \frac{x}{2}$.

When $y = 4$, $16 = 4x - 4x \Rightarrow 16 = 0$ so there are no horizontal tangents.

When $y = \frac{x}{2}$, $\frac{x^2}{4} = x\left(\frac{x}{2}\right) - 4x$.

$$0 = \frac{x^2}{4} - 4x \qquad x = 0 \Rightarrow 4 = 0$$

$$x = 16 \Rightarrow y = 8$$

Vertical tangents occur at $(0,0)$ and $(16,8)$.

Return to Review Topic

12H.1. The point is $(1, -2)$. To find the slope we need to evaluate $f'(1)$.
$f'(x) = 3x^2 + 1$, so that $f'(1) = 3(1)^2 + 1 = 4$.

$$y - (-2) = 4(x - 1) \Rightarrow y + 2 = 4x - 4$$

so that $y = 4x - 6$.

Return to Problem

12H.2. When $x = 2$, $g(2) = \ln(2^2 + 2(2) + 4) = \ln 12$. Thus the point is $(2, \ln 12)$.

$$f'(x) = \frac{1}{x^2 + 2x + 4} \cdot (2x + 2)$$
$$f'(2) = \frac{2 \cdot 2 + 2}{2^2 + 2 \cdot 2 + 4} = \frac{6}{12} = \frac{1}{2}$$

The equation of the line is

$$y - \ln 12 = \frac{1}{2}(x - 2)\text{so that}$$
$$y = \frac{1}{2}x - 1 + \ln 12.$$

Return to Problem

12H.3. The curve crosses the x-axis when $y = 0 \Rightarrow 0 + x = 1 + \cos 0$ so that the point is $(2, 0)$.

We need to use implicit differentiation to find the slope

$$\frac{d}{dx}[y + x = 1 + \cos(xy)]$$
$$y' + 1 = -\sin(xy)(1 + y')$$

Rather than solving for y', plug in the point first.

$$y' + 1 = -\sin(0)(1 + y'),$$

so $y' = -1$.

The equation of the line is

$$y - 0 = -1(x - 2) \quad \text{or} \quad y = -x + 2.$$

Return to Problem

12H.4. To find where the tangent line is horizontal we need to determine where $f' = 0$.

$$\begin{aligned} f' &= 3x^2 - 4x \\ f' = 0 &= 3x^2 - 4x \\ &= x(3x-4). \end{aligned}$$

Thus $x = 0$ and $x = \frac{4}{3}$.

To find the point on the curve we need to evaluate f at $x = 0$ and $x = \frac{4}{3}$.

$$\begin{aligned} f(0) &= 0 - 2 \cdot 0 + 5 = 5 \\ f\left(\frac{4}{3}\right) &= \left(\frac{4}{3}\right)^3 - 2\left(\frac{4}{3}\right)^2 + 5 = \frac{103}{27}. \end{aligned}$$

The tangent line is horizontal at $(0, 5)$ and $\left(\frac{4}{3}, \frac{103}{27}\right)$.

Return to Problem

12H.5. Begin by computing the derivative using implicit differentiation

$$2yy' = 2 \text{ so that } y' = \frac{1}{y}.$$

The tangent line will be vertical when $y = 0$. (Here $y' = \frac{1}{0}$). $y = 0$ when $2x - 6 = 0$ or $x = 3$. If you draw the graph you can see why this is true!

Return to Problem

12H.6. We must first find the equation of the tangent line to the curve $f(x) = \dfrac{1}{x}$ at $x = 2$. At $x = 2$, $f(2) = \dfrac{1}{2}$ and the point is given by $\left(2, \dfrac{1}{2}\right)$.

To find the slope we need to compute $f'(2)$.

$$f' = -\frac{1}{x^2}, \quad f'(2) = -\frac{1}{2^2} = -\frac{1}{4} \text{ and}$$
$$y - \frac{1}{2} = -\frac{1}{4}(x-2)$$
$$y = -\frac{1}{4}x + 1$$

We see that to find a we need to determine where the tangent line crosses the x-axis. At this point $y = 0$ so that $0 = -\dfrac{1}{4}x + 1$ and $x = 4$. Thus, $a = 4$.

Return to Problem