MATH 250 – REVIEW TOPIC 2

Equivalent Forms of Algebraic Expressons

Algebra can be used to change the form of many fractional expressions.

Examples:

Basic simplification

1) $$\frac{2-x}{x^2-4} = \frac{2-x}{(x-2)(x+2)} = \frac{-1}{x+2} = -\frac{1}{x+2}$$

"Rationalizing"

Mult. by 1

2) $$\frac{\sqrt{x}-3}{x-9} = \frac{(\sqrt{x}-3)}{(x-9)} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{x-9}{(x-9)(\sqrt{x}+3)} = \frac{1}{\sqrt{x}+3}$$

3) $$\frac{n+1}{n^2+4n} = \frac{n(1+\frac{1}{n})}{n^2(1+\frac{4}{n^2})} = \frac{1}{n}\left[\frac{1+\frac{1}{n}}{1+\frac{4}{n}}\right]$$ Factoring highest power from numerator and denominator

"Do the examples above bring back old (fond?) memories of LIMITS?"

Separation

4) $$\frac{x^2-3x}{\sqrt{x}} = \frac{x^2}{x^{1/2}} - \frac{3x}{x^{1/2}} = x^{3/2} - 3x^{1/2}$$

5) $$\frac{x^2+1}{x-1} = x+1+\frac{2}{x-1}$$ Long Division

Mult. by 1

6) $$\frac{1}{1+e^{-x}} = \frac{1}{1+e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^x}{e^x+1}$$

Mult. by 1

7) $$\frac{1-\sin x}{\cos x} = \frac{1-\sin x}{\cos x} \cdot \frac{1+\sin x}{1+\sin x} = \frac{1-\sin^2 x}{\cos x(1+\sin x)}$$
$$= \frac{\cos^2 x}{\cos x(1+\sin x)} = \frac{\cos x}{1+\sin x}$$

Before we comment on the algebra, let's examine these examples in a calculus context. Since our virtual calculus site is still in development, you'll need paper and pencil.

Exercise 1: Evaluate the following.

a) $\displaystyle\lim_{x\to 2}\frac{2-x}{x^2-4}$

b) $\displaystyle\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

c) $\displaystyle\lim_{n\to\infty}\frac{n+1}{n^2+4n}$

d) $\displaystyle\int\frac{x^2-3x}{\sqrt{x}}\,dx$

Answers

e) $\displaystyle\int\frac{x^2+1}{x-1}\,dx$

f) $\displaystyle\int\frac{1}{1+e^{-x}}\,dx$

g) $\displaystyle\int\frac{1-\sin x}{\cos x}\,dx$

Answers

Note that algebra is essential in evaluating the above limits and integrals. Another way to evaluate $\displaystyle\int\frac{1-\sin x}{\cos x}$ would be as follows:

$$\int\frac{1-\sin x}{\cos x}\,dx = \int\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)dx = \int(\sec x-\tan x)dx$$
$$= \ln|\sec x+\tan x| - \log|\sec x| + c.$$

Just for fun (we're not afraid to use fun and calculus in the same sentence), show both answers to Exercise 1(g) are equivalent.

A. Long Division

A fraction (rational expression) that has a monomial denominator, like Example 4, easily separates. However, if the fraction is improper (degree of the numerator is greater than or equal to degree of the denominator), then dividing by a polynomial requires Long Division. Here's Example 5 worked out.

$$\begin{array}{r|l} & x + 1 \\ \hline x-1 & x^2 + 1 \\ & \underline{x^2 - x} \\ & \quad x + 1 \\ & \quad \underline{x - 1} \\ & \qquad 2 \end{array}$$

or

$$\frac{(x^2-1)+2}{x-1} = \frac{x^2-1}{x-1} + \frac{2}{x-1} = x + 1 + \frac{2}{x-1}$$

B. Multiplying by "1"

Of our original examples, three of them (#2, 6, and 7) fall into this category. The integral for sec x is derived using multiplication by a form of 1.

$$\int \sec x\,dx = \int \sec x \left(\overbrace{\frac{\sec x + \tan x}{\sec x + \tan x}}^{\text{Mult. by 1}} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

Let $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)dx$

$$= \int \frac{du}{u} = \ln|\sec x + \tan x| + c.$$

C. Completing the Square

In your previous math courses, completing the square was used to solve quadratics or to change the form of a second degree polynomial of the type $Ax^2 + By^2 + Cx + Dy + E = 0$. In Calc II, completing the square will be a useful (and necessary) tool in evaluating certain integrals.

Illustrations: Complete the square.

i. $x^2 - 6x + 4 = (x^2 - 6x + 9) + 4 - 9 = (x - 3)^2 - 5$

ii. $4 - 2x - x^2 = 4 - (2x + x^2) = 4 - (1 + 2x + x^2) + 1 = 5 - (1 + x)^2$

iii. $$\begin{aligned} 2x^2 - 10x = 2(x^2 - 5x) &= 2\left[x^2 - 5x + \left(\frac{5}{2}\right)^2\right] - 2\left(\frac{5}{2}\right)^2 \\ &= 2\left(x - \frac{5}{2}\right)^2 - \frac{25}{2} \end{aligned}$$

Are you following the process? Maybe you need to try this yourself.

Exercise 2: Rewrite each expression by completing the square.

a) $x^2 - 5x + 2$ b) $5 + 4x - x^2$ c) $2x^2 + 12x + 19$

Answers

Now let's try to complete the square starting with the quadratic form $y = ax^2 + bx + c$.

$$\begin{aligned} y &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + c - a\left(\frac{b}{2a}\right)^2 \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \end{aligned}$$

We have completed the square, but what good is this expression? Set $y = 0$

and find x-intercepts:

$$\begin{aligned}\left(x^2+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2}\\ x+\frac{b}{2a} &= \frac{\pm\sqrt{b^2-4ac}}{2a}\\ x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}.\end{aligned}$$

Now you know why math instructors have such a high regard for completing the square.

We will finish this section with examples of two integrals.

Example. $\displaystyle\int \frac{x+2}{x^2+4x+5}\,dx;$

Let $u=x^2+4x+5$, $du=(2x+4)dx \Rightarrow \frac{1}{2}du=(x+2)dx$

$$\begin{aligned}\int \frac{x+2}{x^2+4x+5}\,dx &= \frac{1}{2}\int\frac{du}{u}\\ &= \frac{1}{2}\ln|x^2+4x+5|+c\end{aligned}$$

Example. $\displaystyle\int \frac{1}{x^2+4x+5}\,dx;$

Unlike our first example, substitution won't work. Let's complete the square and see what follows.

$$\begin{aligned}\int \frac{1}{x^2+4x+5}\,dx &= \int\frac{1}{(x+2)^2+1}\,dx\\ &= \arctan(x+2)+c\end{aligned}$$

Recall $\displaystyle\frac{d}{dx}\arctan(\ \ )=\frac{(\ \ )'}{(\ \ )^2+1}$ so $\displaystyle\int\frac{(\ \ )'}{(\ \ )^2+1}\,dx=\arctan(\ \ ).$

Conclusion: This review topic (as well as Topics 3 and 4) demonstrates how algebra is used to evaluate limits and integrals. The challenge is not just knowing the algebra, but being able to recognize when to apply it. That only comes from practice.

Practice Problems.

2.1 Find the quotient and remainder:

a) $\dfrac{x^2-3}{x-2}$ b) $\dfrac{x^3}{x^2+1}$ c) $\dfrac{2x^3+x^2-7x+7}{x^2+x-2}$

Answers

2.2 All of the following problems require multiplication by a form of 1.

a) $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$ Answer

b) Derive $\int \csc x \, dx$ Answer

c) $\int \dfrac{1}{e^{2x}-1} \, dx$ Answer

d) $\int \dfrac{4}{e^{-x}+1} \, dx$ Answer

2.3 Complete each square.

a) x^2-5x+2 Answer

b) $2x^2-12x$ Answer

c) $\sqrt{4-2x-x^2}$ Answer

Answers to Practice Problems.

2.1 a) $x-2\overline{\big)\,x^2-3}$ with quotient $x+2+\dfrac{1}{x-2}$ or $\dfrac{x^2-4+1}{x-2}=\dfrac{x^2-4}{x-2}+\dfrac{1}{x-2}=x+2+\dfrac{1}{x-2}$

Return to Problem

b) $x^2+1\overline{\big)\,x^3}$ with quotient $x-\dfrac{x}{x^2+1}$

Return to Problem

c) $2x-1+\dfrac{-2x+5}{x^2+x-2}$

Return to Problem

2.2 a) $\displaystyle\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}=\lim_{x\to 9}\frac{1}{\sqrt{x}+3}=\frac{1}{6}$

Return to Problem

b)
$$\int \csc x\left(\frac{\csc x+\cot x}{\csc x+\cot x}\right)dx$$
$$=\int\frac{\csc^2 x+\csc x\cot x}{\csc x+\cot x}\,dx \qquad u=\csc x+\cot x,\quad du=(-\csc x\cot x-\csc^2 x)dx$$
$$=-\int\frac{du}{u}$$
$$=-\ln|\csc x+\cot x|+c$$

Return to Problem

2.2 c)
$$\int \frac{1}{e^{2x}-1}\cdot\frac{e^{-2x}}{e^{-2x}}\,dx$$
$$= \int \frac{e^{-2x}}{1-e^{-2x}}\,dx \qquad u = 1-e^{-2x}$$
$$\qquad du = 2e^{-2x}dx$$
$$= \frac{1}{2}\int \frac{du}{u}$$
$$= \frac{1}{2}\ln|1-e^{-2x}| + c$$

Return to Problem

d)
$$\int \frac{4}{e^{-x}+1}\cdot\frac{e^x}{e^x}\,dx = \int \frac{4e^x}{1+e^x}\,dx,$$
$$\text{let } u = 1+e^x$$
$$= 4\ln(1+e^x) + e$$

Return to Problem

2.3 a)
$$\left[x^2 - 5x + \left(\frac{5}{2}\right)^2\right] + 2 - \left(\frac{5}{2}\right)^2 = \left(x-\frac{5}{2}\right)^2 - \frac{17}{4}$$

Return to Problem

b) $2(x-3)^2 - 18$

Return to Problem

c) $\sqrt{4-(2x+x^2)} = \sqrt{4-(1+2x+x^2)+1} = \sqrt{5-(1+x)^2}$

Return to Problem

a) $\lim_{x\to 2} \frac{2-x}{x^2-4}$

b) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

c) $\lim_{n\to\infty} \frac{n+1}{n^2+4n}$

d) $\int \frac{x^2-3x}{\sqrt{x}}\,dx$

Answers

a) $\lim_{x\to 2} \left(\frac{-1}{x+2}\right) = -\frac{1}{4}$

b) $\lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}$

c)
$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1+\frac{1}{n}}{1+\frac{4}{n}}\right]$$
$$= \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{(1+\frac{1}{n})}{(1+\frac{4}{n})}$$
$$= 0 \cdot 1 = 0$$

Recall that for $r > 0$,
$\frac{1}{n^r} \to 0$ as n$\to \infty$
OR
"When denominator grows fraction approaches 0"

Alternate Method:

$$\lim_{n\to\infty} \frac{n+1}{n^2+4n} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{4}{n}} = \frac{0}{1} = 0$$

d) $\frac{2}{5}x^{5/2} - 2x^{3/2} + c$

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Exercise 1 (Continued)

e) $\displaystyle\int \frac{x^2+1}{x-1}\,dx$ f) $\displaystyle\int \frac{1}{1+e^{-x}}\,dx$ g) $\displaystyle\int \frac{1-\sin x}{\cos x}\,dx$

Answers:

e) $\dfrac{1}{2}x^2 + x + 2\ln|x-1| + c$

f) Let $u = e^x + 1$. Why is $\ln|e^x+1| = \ln(e^x+1)$?
Ans: $\ln(e^x+1)+c$.

g) Let $u = 1 + \sin x$.
Ans: $\ln|1+\sin x| + c$.

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a) $x^2 - 5x + 2$ b) $5 + 4x - x^2$ c) $2x^2 + 12x + 19$

Answers:

a) $\left(x - \frac{5}{2}\right)^2 - \frac{17}{4}$ b) $9 - (x - 2)^2$ c) $2(x + 3)^2 + 1$

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