MATH 250 – REVIEW TOPIC 5

Introduction to Series

A significant amount of time in Calc II is devoted to the study of infinite series. This review section will cover introductory material related to series, namely, factorials and summation.

I. Factorials

Exclamation Point: the mark ! after a phrase or command used to indicate strong feeling or excitement.

Examples: No more calculus!
SIU made it to the Final Four!
My tuition got raised 50%!!

Do you suppose $n!$ means something "exciting" is about to happen to n? We'll let you be the judge of that!

In mathematics, $n!$ (read n factorial) is defined as follows:

For any positive integer n, $n \geq 1$,
$$n! = n(n-1)(n-2)\dots 2 \cdot 1.$$
$$0! = 1 \quad \text{by definition}$$

Illustration:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$100! = 100 \cdot 99 \cdot 98 \cdot \ldots \cdot 2 \cdot 1$$
$$(n-1)! = (n-1)(n-2)(n-3)\cdots 2 \cdot 1$$
$$(2n)! = 2n(2n-1)(2n-2)\cdots 2 \cdot 1$$

Exercise 1: What is $n!$, $(2n)!$, $(n-2)!$ if $n = 5, 8$? Answers

An important concept in Calc II is simplifying quotients containing factorials.

Example. $\dfrac{6!}{4!} = \dfrac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1} = 6\cdot 5\left(\dfrac{4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1}\right) = 6\cdot 5$

Let's try something more efficient.

$$\frac{50!}{48!} = \frac{50\cdot 49\cdot 48!}{48!} = 50\cdot 49\left(\frac{48!}{48!}\right) = 50\cdot 49$$

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1)n\cdot(n-1)!}{(n-1)!} = (n+1)n\left(\frac{(n-1)!}{(n-1)!}\right) = (n+1)n$$

In the last two examples, notice how we rewrote the factorial to allow cancellation. This means, depending on the need, we can write

$$\begin{aligned} 5! &\text{ as } 5\cdot 4! \ \text{ or } \ 5\cdot 4\cdot 3!, \\ n! &\text{ as } n(n-1)! \ \text{ or } \ n(n-1)(n-2)!, \ \text{ or} \\ (2n+1)! &\text{ as } (2n+1)(2n)! \end{aligned}$$

Exercise 2: What is $\dfrac{(n+1)!}{n!}, \dfrac{(2(n+1))!}{(2n)!}, \dfrac{(3(n+1))!}{(3n)!}$? Answers

II. Summation

A. Notation

In mathematics, "summation" describes the adding of a list of numbers. We'll start by introducing notation that represents a sum in a concise manner.

Example: Consider $\displaystyle\sum_{i=1}^{10} i^2$.

The letter i is called the index (any letter can be used).

Besides the index, sum notation includes: a) $\sum$, the uppercase greek letter Sigma (for sum), b) upper and lower bounds for the index, and c) an expression to indicate how terms are generated. This expression can be thought of as a function, with integers as inputs and terms in the sum as outputs. To list terms, simply start with the lower bound and substitute integers into the expression, one after another. The last term in the sum corresponds to the substitution of the upper bound.

Thus, $\sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + \cdots + 10^2 = 1 + 4 + 9 + \cdots + 100$.

Additional Examples.

$$\sum_{n=1}^{5} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$$

$$\sum_{n=0}^{100} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \left(\frac{2}{3}\right)^{100}$$

Exercise 3: Write out the following summations.

a) $\sum_{n=1}^{5} \frac{1}{n(n+1)}$ b) $\sum_{k=0}^{10} \frac{1}{2k+1}$ c) $\sum_{n=2}^{6} \frac{2^n}{n!}$ Answers

Warning: Summations may look different, but still be equivalent.

Example: $\sum_{n=1}^{5} \frac{1}{n}$ and $\sum_{n=0}^{4} \frac{1}{n+1}$ are equivalent.
Both yield $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

Example: $\sum_{n=0}^{5} (-1)^n$ and $\sum_{n=0}^{5} \cos n\pi$ are equivalent.
Both yield $1 - 1 + 1 - 1 + 1$.

Exercise 4: Which of the following is equivalent to $\sum_{i=1}^{5} \frac{1}{2i-1}$? (Hint: Write out terms and compare.)

a) $\sum_{n=0}^{4} \frac{1}{2n+1}$ b) $\sum_{k=2}^{6} \frac{1}{2k-3}$ c) $\sum_{n=0}^{4} \frac{(-1)^{2n}}{2n+1}$ Answers

We offer a final comment on notation. It is more than coincidental that sigma notation closely resembles the notation used for definite integrals. The relationship between certain summations (called Riemann Sums) and definite integrals is covered in Calc I.

B. Finding the Sum

If a summation is finite, it's logical to expect that the actual sum can be determined. We could always use technology. Instead, let's develop some useful strategies for finding a sum.

Example: $\sum_{n=1}^{500} e$.
Start by listing the terms.

$$\sum_{n=1}^{500} = \underbrace{e + e + e + \cdots + e}_{\text{(500 terms)}}$$

So $\sum_{n=1}^{100} = 500e$.

Example: $\sum_{n=0}^{10} \cos(2n\pi)$.

$$\begin{aligned}\sum_{n=0}^{10} \cos(2n\pi) &= \overbrace{\cos 0 + \cos 2\pi + \cos 4\pi + \cdots + \cos 20\pi}^{\text{11 terms}} \\ &= 1 + 1 + 1 + \cdots + 1 \\ &= 11\end{aligned}$$

Example: $\sum_{i=1}^{100} i = 1 + 2 + 3 + \cdots + 98 + 99 + 100.$

Try grouping the terms as follows:

$$\underset{1}{(1+100)} + \underset{2}{(2+99)} + \underset{3}{(3+98)} + \ldots + \underset{49}{(49+52)} + \underset{50}{(50+51)}$$

so $\sum_{i=1}^{100} i = 50(101) = 5050.$

Example: $\sum_{n=2}^{10} \frac{1}{n^2 - n}$;

This problem is quite interesting so we'll start with a hint: Use partial fraction decomposition.

$$\frac{1}{n^2 - n} = \frac{1}{n-1} - \frac{1}{n}$$

(Have you looked at Review Topic 3?)

So

$$\sum_{n=2}^{10} \frac{1}{n^2 - n} = \sum_{n=2}^{10} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right).$$

Like our previous example, a rearrangement proves helpful.

$$= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{9} + \frac{1}{9}\right) - \frac{1}{10}$$

So $\sum_{n=2}^{10} \frac{1}{n^2 - n} = \frac{9}{10}$.

Exercise 5: Find each sum.

a) $\sum_{k=1}^{50} (-1)^{k-1} k$ b) $\sum_{n=0}^{20} \sin \frac{(2n+1)\pi}{2}$ c) $\sum_{n=1}^{10} \frac{2}{n(n+2)}$ Answers

C. General Form of a Summation

The general form of a summation can be expressed as:

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots + a_k.$$

Note: just like a_3 is the term following a_2, a_{n+1} is the term that follows a_n.

Here is a problem you will encounter often:

Example: Suppose $a_n = 2n - 1$, find a_{n+1}. Though this merely involves substitution, we have a suggestion.

$$a_{(\ \ )} = 2(\ \ ) - 1 \Rightarrow a_{n+1} = 2(n+1) - 1 = 2n + 1.$$

Exercise 6: Given a_n, find a_{n+1} for

a) $a_n = 3n$ b) $a_n = (n+2)!$ c) $a_n = \dfrac{2^n}{n-1}$ Answers

Taking this one step further:

Example. If $a_n = \dfrac{2^n}{n!}$, find and simplify $\dfrac{a_{n+1}}{a_n}$.

With $a_{n+1} = \dfrac{2^{n+1}}{(n+1)!}$,

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2^{n+1}}{(n+1)!}}{\dfrac{2^n}{n!}} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)n!} = \frac{2}{n+1}.$$

Exercise 7: Given a_n, find and simplify $\dfrac{a_{n+1}}{a_n}$ for the following.

a) $a_n = 3n$ b) $a_n = (n+2)!$ c) $a_n = \dfrac{2^n}{2n-1}$ Answers

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What is $n!$, $(2n)!$, $(n-2)!$ if $n = 5$, 8?

Answers:

$$\begin{aligned} n &= 5 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ 10! &= 10 \cdot 9 \cdot 8 \cdot \ldots \cdot 2 \cdot 1 \\ 3! &= 3 \cdot 2 \cdot 1 \end{aligned}$$

$$\begin{aligned} n &= 8 \\ 8! &= 8 \cdot 7 \cdot 6 \cdot \ldots \cdot 2 \cdot 1 \\ 16! &= 16 \cdot 15 \cdot 14 \cdot \ldots \cdot 2 \cdot 1 \\ 6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

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What is $\dfrac{(n+1)!}{n!}$, $\dfrac{(2(n+1))!}{(2n)!}$, $\dfrac{(3(n+1))!}{(3n)!}$?

Answers:

$$\frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = n+1$$

$$\frac{(2(n+1))!}{(2n)!} = \frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)2n!}{(2n)!} = (2n+2)((2n+1)$$

$$\frac{(3(n+1))!}{(3n)!} = (3n+3)(3n+2)(3n+1)$$

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Write out the following summations.

a) $\displaystyle\sum_{n=1}^{5} \frac{1}{n(n+1)}$ b) $\displaystyle\sum_{k=0}^{10} \frac{1}{2k+1}$ c) $\displaystyle\sum_{n=2}^{6} \frac{2^n}{n!}$

Answers:

a) $\displaystyle \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \frac{1}{5(6)}$

b) $\displaystyle 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{21}$

c) $\displaystyle \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!}$

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Which of the following is equivalent to $\sum_{i=1}^{5} \frac{1}{2i-1}$? Write out terms and compare.

a) $\sum_{n=0}^{4} \frac{1}{2n+1}$ b) $\sum_{k=2}^{6} \frac{1}{2k-3}$ c) $\sum_{n=0}^{4} \frac{(-1)^{2n}}{2n+1}$

Answers:

$$\left.\begin{array}{ll} & \displaystyle\sum_{i=1}^{5} \frac{1}{2i-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \\ \text{a)} & \displaystyle\sum_{n=0}^{4} \frac{1}{2n+1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \\ \text{b)} & \displaystyle\sum_{k=2}^{6} \frac{1}{2k-3} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \\ \text{c)} & \displaystyle\sum_{n=0}^{4} \frac{(-1)^{2n}}{2n+1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \end{array}\right\} \text{all are equivalent}$$

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Find each sum.

a) $\displaystyle\sum_{k=1}^{50}(-1)^{k-1}k$ b) $\displaystyle\sum_{n=0}^{20}\sin\frac{(2n+1)\pi}{2}$ c) $\displaystyle\sum_{n=1}^{10}\frac{2}{n(n+2)}$

Answers:

a)
$$\begin{aligned}\sum_{k=1}^{50}(-1)^{k-1}k &= (1-2)+(3-4)+\cdots-48+(49-50)\\ &= \underbrace{(-1)+(-1)+(-1)\cdots+(-1)}_{\text{25 terms}}\\ &= -25\end{aligned}$$

b)
$$\begin{aligned}\sum_{n=0}^{20}\sin\frac{(2n+1)\pi}{2} &= \sin\frac{\pi}{2}+\sin\frac{3\pi}{2}+\sin\frac{5\pi}{2}+\cdots+\sin\frac{41\pi}{2}\\ &= \underbrace{1+(-1)+1+(-1)\cdots+1}_{\text{21 terms}}\\ &= 1\end{aligned}$$

c) Using Partial Fraction Decomposition (RT3),
$$\frac{2}{n(n+2)} = \frac{A}{n}+\frac{B}{n+2} \Rightarrow 2 = A(n+2)+Bn,$$
so $A=1,\ B=-1$.

Thus
$$\begin{aligned}\sum_{n=1}^{10}\frac{2}{n(n+2)} &= \sum_{n=1}^{10}\left(\frac{1}{n}-\frac{1}{n+2}\right) = \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)\\ &\qquad +\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{9}-\frac{1}{11}\right)+\left(\frac{1}{10}-\frac{1}{12}\right)\\ &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{10}\\ &\qquad -\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{10}-\frac{1}{11}-\frac{1}{12}\\ &= 1+\frac{1}{2}-\frac{1}{11}-\frac{1}{12} = \frac{175}{132}\end{aligned}$$

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Given a_n, find a_{n+1} for

a) $a_n = 3n$ b) $a_n = (n+2)!$ c) $a_n = \dfrac{2^n}{n-1}$

Answers:

a) $a_{(\ )} = 3(\ )$; $a_{n+1} = 3(n+1) = 3n+3$

b) $a_{(\ )} = [(\ )+2]!$; $a_{n+1} = [(n+1)+2]! = (n+3)!$

c) $a_{(\ )} = \dfrac{2^{(\ )}}{(\ )-1}$; $a_{n+1} = \dfrac{2^{n+1}}{n}$

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Given a_n, find and simplify $\dfrac{a_{n+1}}{a_n}$.

a) $a_n = 3n$ b) $a_n = (n+2)!$ c) $a_n = \dfrac{2^n}{2n-1}$

Answers:

a) $$\frac{a_{n+1}}{a_n} = \frac{3n+3}{3n} = \frac{n+1}{n}$$

b) $$\frac{a_{n+1}}{a_n} = \frac{(n+3)!}{(n+2)!} = \frac{(n+3)(n+2)!}{(n+2)!} = n+3$$

c) $$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2n+1} \cdot \frac{2n-1}{2^n} = 2\left(\frac{2n-1}{2n+1}\right)$$

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