MATH 250 – REVIEW TOPIC 8

Polar Graphs

This section presents a quick introduction to polar graphs. This topic will be covered in greater detail in Calc II.

Polar graphs are curves plotted in polar coordinates (the previous review topic). The curves arise from equations written in terms of the polar coordinates (r, θ).

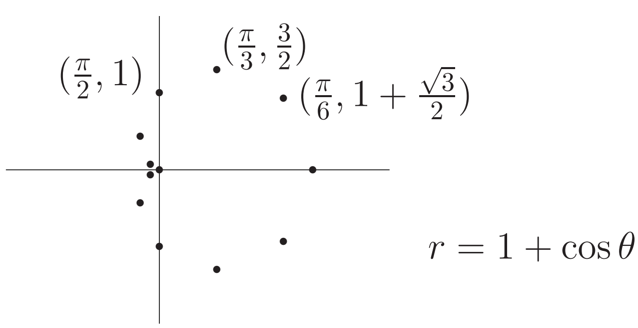
Example 8.1. Graph the curve r= $1 + \cos\theta$. (8.1)

One way to proceed is to make a table like the one below. Set $\theta =$ "certain values" (usually the values where $\cos\theta$ can be evaluated exactly) and solve for r.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	2	$1+\frac{\sqrt{3}}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$1-\frac{\sqrt{3}}{2}$	0	$1-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$1+\frac{\sqrt{3}}{2}$	2

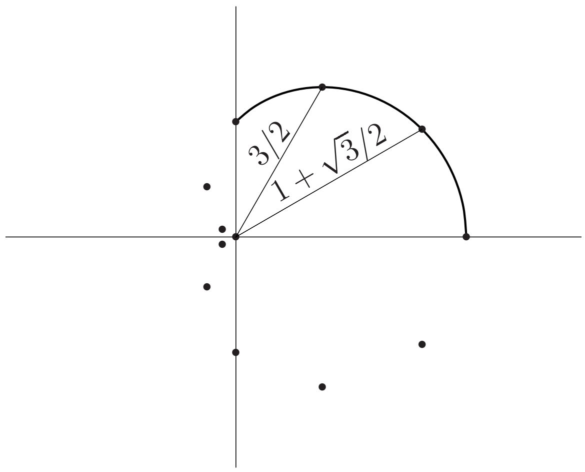
Table 8.1.

Since $\cos\theta$ is periodic of period 2π, plugging in values of $\theta < 0$ or $\theta > 2\pi$ will repeat the same values of r. We could have chosen additional values of θ between 0 and 2π, say $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{8}$ (use your calculator), etc. Knowing what values of θ to choose is best learned by experience. Plotting the points in Table 8.1 gives the following.



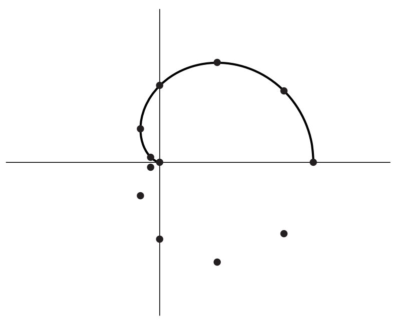
How do we connect the points? Looking at the curve $r = 1 + \cos\theta$ and Table 8.1, we see that as θ goes from 0 to $\frac{\pi}{2}$, r decreases from 2 to 1. (This means that as θ

increases, the value of r on each ray is getting smaller.) Thus we have



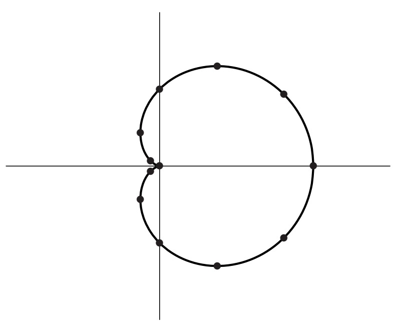
$r = 1 + \cos\theta$

Similarly, as θ goes from $\dfrac{\pi}{2}$ to π, r decreases from 1 to 0. This gives



$r = 1 + \cos\theta$

As θ goes from π to 2π, r is increasing from 0 to 2 and we obtain the rest of the curve, called a cardioid (Fig. 8.1).



$r = 1 + \cos\theta$

Fig. 8.1.

If we let θ range from 0 to 4π, we would trace the curve twice. In Calc II you will learn other techniques (like symmetry) to help you plot polar curves.

Notice if we had written this equation $r = 1 + \cos\theta$ in rectangular form, we would have the equation $x^2 + y^2 = \sqrt{x^2 + y^2} + x$. Imagine trying to plot that in rectangular coordinates!

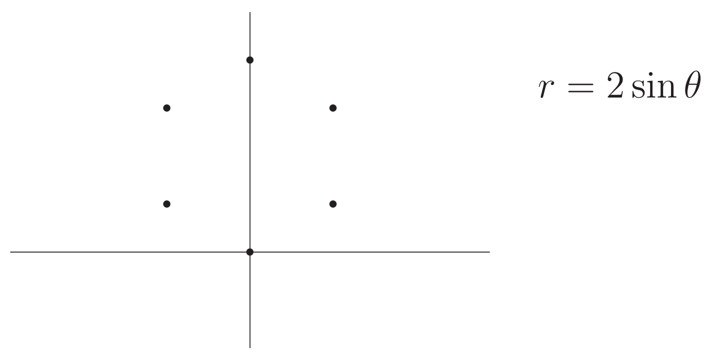
Example 8.2. Plot $r = 2\sin\theta$. (8.2)

Making a table yields the following.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	0	1	$\sqrt{3}$	2	$\sqrt{3}$	$\frac{1}{2}$	0	-1	$-\sqrt{3}$	-2	$-\sqrt{3}$	-1	0

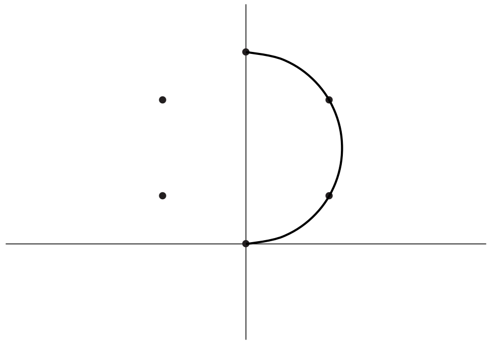
Table 8.2.

Since $\sin\theta$ is periodic of period 2π, it is not necessary to plot any values of θ where $\theta < 0$ or $\theta > 2\pi$. First plot the points in Table 8.2 for θ between 0 and π.



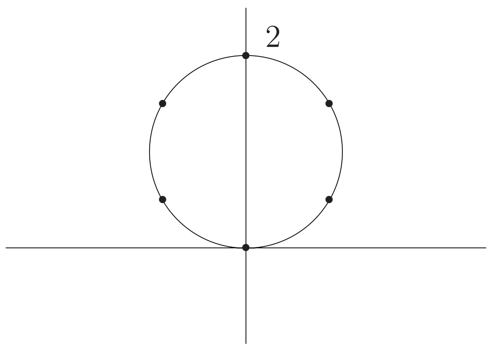
Again, how do we connect the dots? Notice that in the equation $r = 2\sin\theta$, as θ

goes from 0 to $\frac{\pi}{2}$, r goes from 0 to 2. So we have



$r = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{2}$

As θ goes from $\frac{\pi}{2}$ to π, r decreases from 2 to 0.



$r = 2\sin\theta,\ 0 \le \theta \le \pi$

Fig. 8.2.

As θ ranges from 0 to π in Eqn. (8.2), the above figure is formed. If θ ranges from π to 2π, points are repeated. This is because $(-1, \frac{7\pi}{6})$ is the same as $(1, \frac{\pi}{6})$, $(-\sqrt{3}, \frac{4\pi}{3})$ is the same as $(\sqrt{3}, \frac{\pi}{3})$, etc. Thus, the figure is traversed again.

The graph in Fig. 8.2 looks like a circle. To verify this, we can write this equation in rectangular form by multiplying both sides of Eqn. (8.2) by r. This gives

$$r^2 = 2r\sin\theta.$$

Using Eqns (7.2) and (7.3) in Review Topic 7, the above equation can be rewritten as

$$x^2 + y^2 = 2y.$$

Completing the square (Review Topic 2) yields

$$x^2 + y^2 - 2y + 1 = 1, \quad \text{or}$$
$$x^2 + (y-1)^2 = 1.$$

Thus, $r^2 = 2r\sin\theta$ in polar coordinates gives the same graph as $x^2 + (y-1)^2 = 1$ in rectangular coordinates, which is a circle of radius 1 centered at $(0, 1)$.

Example 8.3. Graph the curve $r = 2$.

This problem is so simple it's hard! The angle θ is not specified, so θ can have any value. This means that points of the form $(2, \theta\ )$ satisfy the equation, where θ is any angle. This leads to the graph below, a circle of radius 2 centered at the origin.

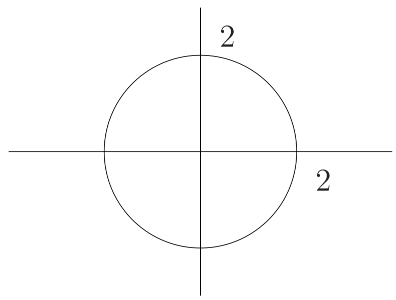


Fig. 8.3

In rectangular coordinates, the above circle has the equation $x^2 + y^2 = 4$. It is much simpler to write $r = 2$. As mentioned in Review Topic 7, this is one of the advantages of polar coordinates.

Exercise 8.1.

(a) Graph the polar curve $r = \cos\theta$. Answer

(b) Graph the region determined by the conditions $1 \le r \le 2$, $\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2}$. Answer

(c) Graph the cardiod $r = 1 - \sin\theta$. Answer

Example 8.4. Consider the curves $r = 1 + \cos\theta$ and $r = 2\sin\theta$. Find any points of intersection (in polar coordinates).

When finding the points of intersection of polar curves, it's a good idea to sketch the curves. From Examples 8.1 and 8.2 we have the following.

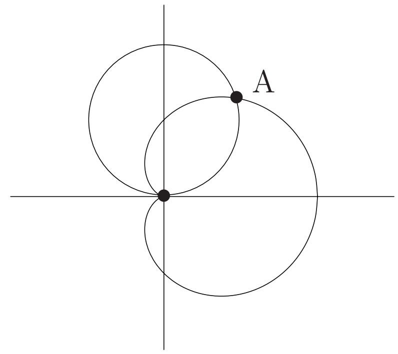


Fig. 8.5

We see that there are two points of intersection, A and the origin or pole. The curve $r = 1 + \cos\theta$ passes through the pole when $\theta = \pi$. That is, $(\pi, 0)$ satisfies $r = 1 + \cos\theta$. It also satisfies $r = 2\sin\theta$. So, $(\pi, 0)$ is one point of intersection. To solve for A, since the curves must have the same value of r at A, we set the curves equal to each other and solve for θ. That is,

$$2\sin\theta = 1 + \cos\theta \Rightarrow \theta = \, ?$$

Finding θ is a bit tricky. Since θ is in the first quadrant, $\cos\theta$ is positive. Thus $\cos\theta = \sqrt{1 - \sin^2\theta}$. This implies

$$\begin{aligned} 2\sin\theta &= 1 + \sqrt{1 - \sin^2\theta} \\ 2\sin\theta - 1 &= \sqrt{1 - \sin^2\theta} \\ 4\sin^2\theta - 4\sin\theta + 1 &= 1 - \sin^2\theta \\ 5\sin^2\theta - 4\sin\theta &= 0 \\ \sin\theta(5\sin\theta - 4) &= 0 \end{aligned}$$

$$\begin{array}{r|l} \sin\theta = 0 & 5\sin\theta = 4 \\ \Rightarrow \theta = 0 \text{ or } \pi & \sin\theta = \dfrac{4}{5} \\ & \theta = \arcsin\dfrac{4}{5} \end{array}$$

Neither of the values 0 or π leads to A. (Remember we've already found π.) Thus $\theta = \arcsin\dfrac{4}{5}$ is the answer. As a check, let's plug $\theta = \arcsin\dfrac{4}{5}$ into both equations and calculate r. (r should be the same for both equations!) For Eqn. (8.2), the statement $\theta = \arcsin\frac{4}{5}$ implies $\sin\theta = \frac{4}{5}$ and leads to the picture below. Thus, we have $r = 2\sin(\arcsin\frac{4}{5}) = 2\sin\theta = 2(\frac{4}{5}) = \frac{8}{5}$. (See Math 150, Review Topic 15(III), for evaluating $\sin(\arcsin\theta)$.)

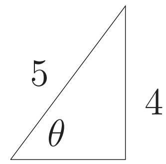


Fig. 8.5

By the Pythagorean Theorem, the missing side in Fig. 8.5 has length 3. Thus $\cos(\arcsin\frac{4}{5}) = \cos(\theta) = \frac{3}{5}$. For Eqn. (8.1) then,

$$r = 1 + \cos(\arcsin\frac{4}{5}) = 1 + \frac{3}{5} = \frac{8}{5}.$$

This means $A = (\frac{8}{5}, \arcsin\frac{4}{5})$, and we have verified that A satisfies both curves.

PRACTICE PROBLEMS for Topic 8 – (Polar Graphs)

8.1. (a) Graph the polar equations $r = 1 - \cos\theta$ and $r = \cos\theta$. Answer

(b) Find the points of intersection. Answer

(c) Shade in the area common to both curves (the area of intersection). Answer

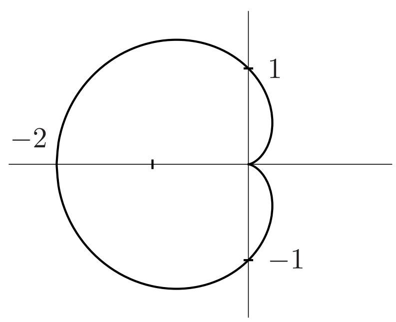
(d) Shade in the area outside $r = \cos\theta$ and inside $r = 1 - \cos\theta$. Answer

ANSWERS to PRACTICE PROBLEMS (Topic 8–Polar Graphs)

8.1. a) The graph of $r = \cos\theta$ is the circle given in Exercise 8.1a). To graph $r = 1 - \cos\theta$, we first make a table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	0	$1-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$1+\frac{\sqrt{3}}{2}$	2	$1+\frac{\sqrt{3}}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$1-\frac{\sqrt{3}}{2}$	0

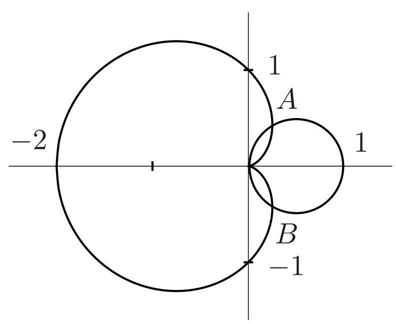
Plotting the above points gives the following graph.



$r = 1 - \cos\theta$

Return to Problem

8.1. b) Now plot $r = \cos\theta$ and $r = 1 - \cos\theta$ on the same set of axes.



We see that the curves intersect at three points; A, B, and the pole (origin). To find A and B, set the r values equal and solve for θ.

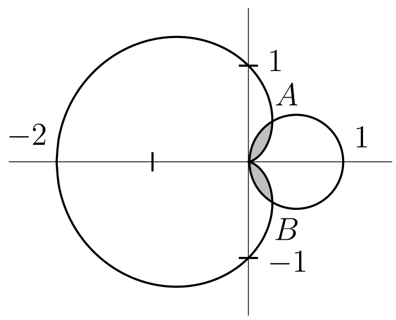
$$\cos\theta = 1 - \cos\theta$$
$$2\cos\theta = 1$$
$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3},\ \frac{5\pi}{3}.$$

If $\theta = \frac{\pi}{3}$ $r = \frac{1}{2}$ and if $\theta = \frac{5\pi}{3}$ $r = \frac{1}{2}$.

So, the points of intersection are the origin, $(\frac{1}{2}, \frac{\pi}{3})$, $(\frac{1}{2}, \frac{5\pi}{3})$.

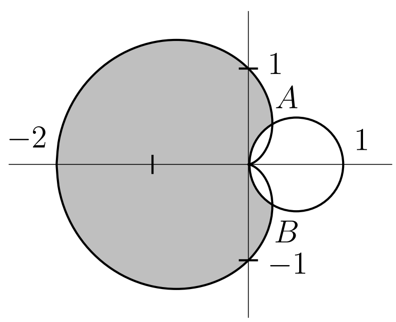
Return to Problem

8.1. c) The area of intersection is shaded.



Return to Problem

8.1. d) The area outside $r = \cos\theta$ and inside $r = 1 - \cos\theta$ is shaded.



Return to Problem

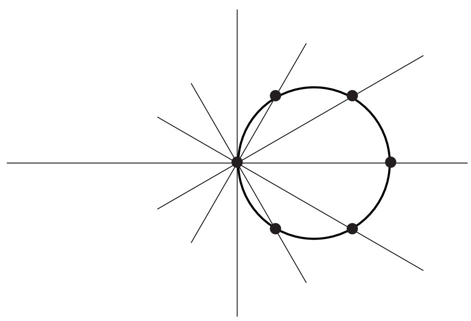
a) Graph the polar curve r= $\cos\theta$.

Answer:

a) To graph the polar curve $r = \cos\theta$, first make a table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Plotting these points yields



To verify the graph is a circle, multiply the equation by r.

$$\Rightarrow r^2 = r\cos\theta \Rightarrow x^2 + y^2 = x \Rightarrow x^2 - x + y^2 = 0$$
$$\Rightarrow x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$$

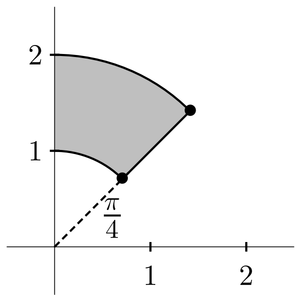
Thus, the graph is a circle centered at $\left(\frac{1}{2}, 0\right)$ of radius $-\frac{1}{2}$. (The solution to Example 8.2 is very similar to the solution for this problem.)

Return to Review Topic

b) Graph the region determined by the conditions $1 \le r \le 2$, $\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2}$.

Answer:

b)



$$1 \le r \le 2; \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$

The conditions above
describe the shaded region.

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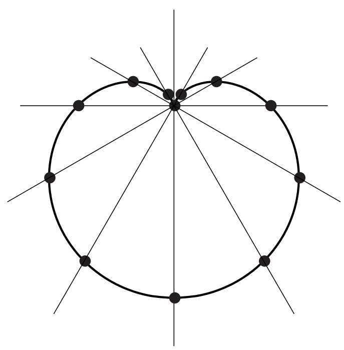
(c) Graph the cardiod $r = 1 - \sin\theta$.

Answer:

c) To graph the cardioid $r = 1 - \sin\theta$, we again make a table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	1	$\frac{1}{2}$	$1 - \frac{\sqrt{3}}{2}$	0	$1 - \frac{\sqrt{3}}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$1 + \frac{\sqrt{3}}{2}$	2	$1 + \frac{\sqrt{3}}{2}$	$\frac{3}{2}$	1

Plotting the polar points above gives the following picture.



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