Some New Results on Lyapunov-Type Diagonal Stability

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Outline

Background and Preliminaries

- Consider the first-order linear constant coefficient system of n ordinary differential equations:

$$\frac{dx}{dt} = A[x(t) - \hat{x}] \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $x(t), \hat{x} \in \mathbb{R}^n$.

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- $\hat{x}$ is called an equilibrium for this system. If $x(t)$ converges to $\hat{x}$ as $t \to \infty$ for every choice of the initial data $x(0)$, the equilibrium $\hat{x}$ is said to be asymptotically stable.

- The equilibrium is asymptotically stable if and only if each eigenvalue of A has a negative real part. A matrix A satisfying this condition is called a (Hurwitz) stable matrix.

Definition

Let $A \in \mathbb{R}^{n\times n}$ be symmetric matrix. Then A is said to be positive semidefinite (positive definite) if $x^*Ax \geq 0$ ($x^*Ax > 0$) for all nonzero $x \in \mathbb{R}^n$.

- A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite (positive definite) if and only if all of its eigenvalues are nonnegative (positive).
- A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite (positive definite) if and only if all its principal minors are nonnegative (positive).
  - The determinant of a principal submatrix is called a principal minor.
- We shall denote $A \succeq 0$ ($A \succ 0$) when A is positive semidefinite (positive definite).

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Lyapunov's Theorem

A matrix $A \in \mathbb{R}^{n \times n}$ is stable if and only if there exists a $P \succ 0$ such that

$$PA + A^T P \succ 0. \tag{2}$$

Then, P is called a Lyapunov solution of (2).

Definition

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be (Lyapunov) diagonally stable if there exists a positive diagonal matrix D such that

$$DA + A^T D \succ 0. \tag{3}$$

Then, D is called a diagonal (Lyapunov) solution of (3).

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Definition

Let $A^{(1)}, A^{(2)}, \ldots, A^{(m)} \in \mathbb{R}^{n \times n}$. If there exists a positive diagonal matrix D such that

$$DA^{(k)} + (A^{(k)})^T D \succ 0, \;\; k = 1, 2, \ldots, m, \tag{4}$$

then D is called a common diagonal (Lyapunov) solution of (4). The existence of such a D is interpreted as the simultaneous diagonal stability of $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$.

- Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The matrix A is stable, having the eigenvalues $1 \pm i$.
- Choosing positive diagonal matrix $D = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}$, we have

$$\begin{aligned} DA + A^T D &= \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \succ 0, \end{aligned}$$

thus A is a diagonally stable matrix.

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- Let $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$. The matrix B is stable, having the eigenvalues $1 \pm i$. However, B is not a diagonally stable matrix.

$$DB + B^T D = \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d_1 & \\ & d_2 \end{bmatrix} = \begin{bmatrix} 4d_1 & 2d_2 - d_1 \\ 2d_2 - d_1 & 0 \end{bmatrix} \not\succ 0.$$

Applications of diagonal stability

- Dynamic models for biochemical reactions
- Systems theory
- Population dynamics
- Communication networks
- Mathematical economics

Applications of simultaneous diagonal stability

- Large-scale dynamic systems
- Interconnected time-varying and switched systems

A Necessary and Sufficient Condition Based on Schur Complement

- We shall denote $\langle k \rangle = \{1, 2, \ldots, k\}$. For $A \in \mathbb{R}^{n \times n}$, let $A[\alpha, \beta]$ be the submatrix of A whose rows and columns are indexed by $\alpha, \beta \subseteq \langle n \rangle$, respectively, and let $A[\alpha] = A[\alpha, \alpha]$.

- The Schur complement of $A[\alpha]$ in A is defined as

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c], \quad (5)$$

where $\alpha^c = \langle n \rangle \backslash \alpha$, provided that $A[\alpha]$ is nonsingular.

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where $\alpha^c = \langle n \rangle \backslash \alpha$, provided that $A[\alpha]$ is nonsingular.

- Consider, for example, the partitioned matrix $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$, where $A[\alpha] = B$, $A[\alpha^c] = E$, $A[\alpha^c, \alpha] = D$, and $A[\alpha, \alpha^c] = C$. Then,

$$A/A[\alpha] = E - DB^{-1}C.$$

Theorem (Redheffer, 1985)

Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix with $A[\{n\}] > 0$ and $\alpha = \langle n-1 \rangle$. Then, A is diagonally stable if and only if $A[\alpha]$ and $A^{-1}[\alpha]$ have a common diagonal solution.

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Theorem (Shorten and Narendra, 2009)

Let $A \in \mathbb{R}^{n\times n}$ be partitioned as $A = \begin{bmatrix} \hat{A} & p \\ q^T & r \end{bmatrix}$, where $\hat{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $r > 0$. Then, A is diagonally stable if and only if $\hat{A}$ and $\hat{A} - \dfrac{pq^T}{r}$ have a common diagonal solution.

- $\left(\hat{A} - \dfrac{pq^T}{r}\right)^{-1} = A^{-1}[\langle n-1\rangle]$

Theorem

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Then, A is diagonally stable with a diagonal solution $D = \begin{bmatrix} \hat{D} & \\ & x \end{bmatrix}$,

where $\hat{D} \in \mathbb{R}^{(n-1)\times(n-1)}$, if and only if the following are true:

(i) $r > 0$.

(ii) $\hat{A}$ and the Schur complement $A/A[\{n\}] = \hat{A} - \dfrac{pq^T}{r}$ share a common diagonal solution $\hat{D}$.

Lemma (Horn and Johnson, 1985)

Suppose that $B \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $\alpha \subset \langle n \rangle$. Then, $B \succ 0$ if and only if

$$B[\alpha] \succ 0$$

and

$$B/B[\alpha] \succ 0.$$

Slyvester's Determinant Theorem

Let $U \in \mathbb{R}^{n \times m}$ and $V \in \mathbb{R}^{m \times n}$. Then

$$\det(I_n + UV) = \det(I_m + VU),$$

where I_k is the $k \times k$ identity matrix.

Proof of Theorem: We need to justify that, for some $x > 0$,

$$B = DA + A^T D = \begin{bmatrix} \hat{D}\hat{A} + \hat{A}^T \hat{D} & \hat{D}p + xq \\ p^T \hat{D} + xq^T & 2xr \end{bmatrix} \succ 0.$$

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This, by lemma with $\alpha = \langle n-1 \rangle$ and $M = B[\alpha] = \hat{D}\hat{A} + \hat{A}^T \hat{D} \succ 0$, is equivalent to that for some $x > 0$,

$$f(x) = B/B[\alpha] = 2xr - (p^T \hat{D} + xq^T) M^{-1} (\hat{D}p + xq) > 0. \qquad (6)$$

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From (6), $f(x) \leq 0$ whenever $x \leq 0$. On the other hand,

$$f(x) = -x^2 q^T M^{-1} q - 2x(q^T M^{-1} \hat{D} p - r) - p^T \hat{D} M^{-1} \hat{D} p.$$

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It suffices to show

$$\Delta = (q^T M^{-1} \hat{D}p - r)^2 - (q^T M^{-1} q)(p^T \hat{D} M^{-1} \hat{D}p) > 0.$$

Hence, we calculate

$$\Delta = \det \begin{bmatrix} -r + q^T M^{-1} \hat{D} p & q^T M^{-1} q \\ p^T \hat{D} M^{-1} \hat{D} p & -r + p^T \hat{D} M^{-1} q \end{bmatrix}$$

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$$= r^2 \det \left(I_2 - \begin{bmatrix} r^{-1} & \\ & r^{-1} \end{bmatrix} \begin{bmatrix} q^T \\ p^T \hat{D} \end{bmatrix} \begin{bmatrix} M^{-1} \hat{D} p & M^{-1} q \end{bmatrix} \right).$$

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By Sylvester's determinant theorem, we have

$$\Delta = r^2 \det \left(I_{n-1} - \begin{bmatrix} M^{-1} \hat{D} p & M^{-1} q \end{bmatrix} \begin{bmatrix} r^{-1} & \\ & r^{-1} \end{bmatrix} \begin{bmatrix} q^T \\ p^T \hat{D} \end{bmatrix} \right).$$

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Continuing with the above, we finally arrive at

$$\Delta = r^2 \det(M^{-1}) \det(\hat{D} S + S^T \hat{D}) > 0,$$

where $S = A/A[\{n\}]$. □

We may specify all the feasible positive $D[\{n\}] = x$ values in a diagonal solution $D = \begin{bmatrix} \hat{D} & \\ & x \end{bmatrix}$ as follows:

- x is in, but does not exceed, $0 \leq x_1 < x < x_2 \leq \infty$, where

$$x_1 = \frac{p^T \hat{D} M^{-1} \hat{D} p}{\sqrt{\Delta} - (q^T M^{-1} \hat{D} p - r)}$$

and

$$x_2 = \frac{\sqrt{\Delta} - (q^T M^{-1} \hat{D} p - r)}{q^T M^{-1} q},$$

with

$$M = \hat{D}\hat{A} + \hat{A}^T \hat{D}$$

and

$$\Delta = (q^T M^{-1} \hat{D} p - r)^2 - (q^T M^{-1} q)(p^T \hat{D} M^{-1} \hat{D} p).$$

In particular, when $q = 0$, $x_1 = \dfrac{p^T \hat{D} M^{-1} \hat{D} p}{2r}$ and $x_2 = \infty$.

Corollary 1

Let $A \in \mathbb{R}^{n \times n}$ and $\alpha = \langle n \rangle \backslash \{k\}$ for some $1 \le k \le n$. Then, A is diagonally stable matrix that has a diagonal solution D with $D[\alpha] = \hat{D}$ and $D[\{k\}] = x$ if and only if the following are true:

- **(i)** $A[\{k\}] > 0$.
- **(ii)** $A[\alpha]$ and the Schur complement $A/A[\{k\}]$ share a common diagonal solution $\hat{D}$.

- The diagonal stability of a matrix A is preserved under simultaneous row and column permutations on A.
- If a matrix A is diagonally stable, then any Schur complement $A/A[\alpha]$ is also diagonally stable for any $\alpha \subseteq \langle n \rangle$.

Corollary 2

Let $A^{(1)}, A^{(2)}, \ldots, A^{(m)} \in \mathbb{R}^{n\times n}$ be each partitioned as $A^{(k)} = \begin{bmatrix} \hat{A}^{(k)} & p^{(k)} \\ (q^{(k)})^T & r^{(k)} \end{bmatrix}$, where $\hat{A}^{(k)} \in \mathbb{R}^{(n-1)\times(n-1)}$. Then $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$ have a common diagonal solution in the form $D = \begin{bmatrix} \hat{D} & \\ & x \end{bmatrix}$, with $\hat{D} \in \mathbb{R}^{(n-1)\times(n-1)}$, if and only if the following are true:

- **(i)** $r^{(k)} > 0, \ k = 1, 2, \ldots, m$.
- **(ii)** $\hat{A}^{(k)}$ and $A^{(k)}/A^{(k)}[\{n\}]$, $k = 1, 2, \ldots, m$, have $\hat{D}$ as a common diagonal solution.
- **(iii)** $x_1 < x_2$, where $x_1 = \max\limits_{1\le k\le m} x_1^{(k)}$, $x_2 = \min\limits_{1\le k\le m} x_2^{(k)}$, and where for each k, $0 \le x_1^{(k)} < x_2^{(k)} \le \infty$ are such that

$$x_1^{(k)} = \frac{(p^{(k)})^T \hat{D}(M^{(k)})^{-1}\hat{D}p^{(k)}}{\sqrt{\Delta^{(k)}} - ((q^{(k)})^T(M^{(k)})^{-1}\hat{D}p^{(k)} - r^{(k)})}$$

and

$$x_2^{(k)} = \frac{\sqrt{\Delta^{(k)}} - ((q^{(k)})^T(M^{(k)})^{-1}\hat{D}p^{(k)} - r^{(k)})}{(q^{(k)})^T(M^{(k)})^{-1}q^{(k)}},$$

with

$$M^{(k)} = \hat{D}\hat{A}^{(k)} + (\hat{A}^{(k)})^T\hat{D}.$$

Corollary 3

For $k = 1, 2, \ldots, m$, let $A^{(k)} = [a_{i,j}^{(k)}] \in \mathbb{R}^{2\times 2}$. Then, $A^{(1)}$, $A^{(2)}$, …, $A^{(m)}$ have a common diagonal solution $D = \begin{bmatrix} 1 & \\ & x \end{bmatrix}$ if and only if the following hold:

(i) $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$ are all P-matrices.

(ii) $x_1 < x_2$, where $x_1 = \max_{1\le k\le m} x_1^{(k)}$, $x_2 = \min_{1\le k\le m} x_2^{(k)}$, and where for each k, $0 \le x_1^{(k)} < x_2^{(k)} \le \infty$ are such that

$$x_1^{(k)} = \left(\frac{a_{1,2}^{(k)}}{\sqrt{a_{1,1}^{(k)} a_{2,2}^{(k)}} + \sqrt{\det(A^{(k)})}} \right)^2$$

and

$$x_2^{(k)} = \left(\frac{\sqrt{a_{1,1}^{(k)} a_{2,2}^{(k)}} + \sqrt{\det(A^{(k)})}}{a_{2,1}^{(k)}} \right)^2 .$$

Example

$$A_1 = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 1 & -3 \\ -4 & 3 & 4 \end{bmatrix},\ A_2 = \begin{bmatrix} 4 & 4 & -1 \\ -2 & 4 & 2 \\ 0 & 3 & 2 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & -3 & 2 \\ 6 & 2 & -1 \\ -6 & -1 & 3 \end{bmatrix}.$$

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- Taking $\alpha = \langle 2 \rangle$, we obtain from Corollary 3 that $A_1[\alpha]$, $A_1/A_1[\alpha^c]$, $A_2[\alpha]$, $A_2/A_2[\alpha^c]$, $A_3[\alpha]$, and $A_3/A_3[\alpha^c]$ have a common diagonal solution $\hat{D} = \begin{bmatrix} 1 & \\ & x \end{bmatrix}$, where $0.877 \approx \dfrac{121}{4\left(2+\sqrt{15}\right)^2} < x < \dfrac{\left(\sqrt{2}+2\sqrt{5}\right)^2}{36} \approx 0.962$.

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- If we choose, for example, $x = 0.9$ and assume that $D = \begin{bmatrix} \hat{D} & \\ & y \end{bmatrix}$, then we can apply Corollary 2 on A_1, A_2, and A_3 to determine that

$$0.604 \approx \frac{1026}{1393+\sqrt{93649}} < y < \frac{1347+6\sqrt{6389}}{2570} \approx 0.71.$$

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$$0.604 \approx \frac{1026}{1393 + \sqrt{93649}} < y < \frac{1347 + 6\sqrt{6389}}{2570} \approx 0.71.$$

- Hence, given any y in the above range, A_1, A_2, and A_3 share a common diagonal solution in the form $D = \begin{bmatrix} 1 & & \\ & 0.9 & \\ & & y \end{bmatrix}$.

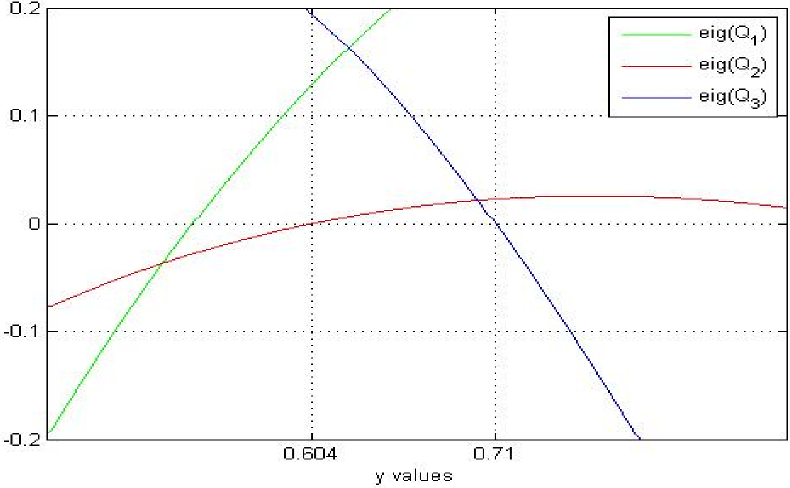

Figure 1: Change in the smallest eigenvalue of $Q_i = DA_i + A_i^T D$, i=1,2,3, depending on y, the last diagonal entry of D.

A New Characterization for Common Diagonal Solutions

Theorem (Barker, Berman and Plemmons, 1978)

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonally stable if and only if for every nonzero $X \succeq 0$, AX has a positive diagonal entry.

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Theorem (Berman, Goldberg and Shorten, 2014)

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$. Then, $\mathcal{A}$ has a common diagonal solution if and only if for any $X^{(k)} \succeq 0$, $k = 1, 2, \ldots, m$, not all of them zero, $\sum_{k=1}^{m} A^{(k)} X^{(k)}$ has a positive diagonal entry.

Theorem (Kraaijevanger, 1991)

The following statements are equivalent for a matrix $A \in \mathbb{R}^{n \times n}$:

(i) A is diagonally stable.

(ii) $A \circ S$ is a P-matrix for all $S \succeq 0$ with diagonal entries all being 1.

(iii) A has positive diagonal entries and $\det(A \circ S) > 0$ for all $S \succeq 0$ with diagonal entries all being 1.

- Hadamard product of two matrices $A = [a_{i,j}] \in \mathbb{R}^{n \times n}$ and $B = [b_{i,j}] \in \mathbb{R}^{n \times n}$ is the matrix $A \circ B = [a_{i,j} b_{i,j}] \in \mathbb{R}^{n \times n}$.
- A matrix A is called a P-matrix (P_0-matrix) if all its principal minors are positive (nonnegative).

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(ii) $A \circ S$ is a P-matrix for all $S \succeq 0$ with diagonal entries all being 1.

(iii) A has positive diagonal entries and $\det(A \circ S) > 0$ for all $S \succeq 0$ with diagonal entries all being 1.

- Hadamard product of two matrices $A = [a_{i,j}] \in \mathbb{R}^{n\times n}$ and $B = [b_{i,j}] \in \mathbb{R}^{n\times n}$ is the matrix $A \circ B = [a_{i,j} b_{i,j}] \in \mathbb{R}^{n\times n}$.
- A matrix A is called a P-matrix (P_0-matrix) if all its principal minors are positive (nonnegative).
- We shall extend Kraaijevanger's result to a new characterization for a set of matrices to share a common diagonal solution.
- Accordingly, we shall extend P-matrices by introducing a new notion called $\mathcal{P}$-sets.

Lemma (Fiedler and Ptak, 1962)

Let $A \in \mathbb{R}^{n \times n}$. Then, A is a P-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, $x_i(Ax)_i > 0$ for some index i.

Lemma (Fiedler and Ptak, 1962)

Let $A \in \mathbb{R}^{n \times n}$. Then, A is a P-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, $x_i(Ax)_i > 0$ for some index i.

Definition

Given $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$, we define $\mathcal{A}$ as a $\mathcal{P}$-set if for any $x^{(k)} \in \mathbb{R}^n$, $k = 1, 2, \ldots, m$, not all of them zero, there exists some index i such that $\sum_{k=1}^{m} x_i^{(k)}(A^{(k)}x^{(k)})_i > 0$.

Lemma (Fiedler and Ptak, 1962)

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Theorem

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$. Then, $\mathcal{A}$ is a $\mathcal{P}$-set if and only if for any $x^{(k)} \in \mathbb{R}^n$, $k = 1, 2, \ldots, m$, not all of them zero, $\sum_{k=1}^{m} A^{(k)}x^{(k)}(x^{(k)})^T$ has a positive diagonal entry.

- If $\mathcal{A}$ has a common diagonal solution, then it is a $\mathcal{P}$-set.

Main Theorem-1

Given $\mathcal{A} = \{A^{(1)}, A^{(2)}, \dots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$, the following are equivalent:

(i) $\mathcal{A}$ has a common diagonal solution.

(ii) $\{A^{(1)} \circ S^{(1)}, A^{(2)} \circ S^{(2)}, \dots, A^{(m)} \circ S^{(m)}\}$ has a common diagonal solution for all $S^{(k)} \succeq 0,\ k = 1, 2, \dots, m$, each with diagonal entries being all 1.

(iii) $\{A^{(1)} \circ S^{(1)}, A^{(2)} \circ S^{(2)}, \dots, A^{(m)} \circ S^{(m)}\}$ is a $\mathcal{P}$-set for all $S^{(k)} \succeq 0,\ k = 1, 2, \dots, m$, each with diagonal entries being all 1.

Outline of the proof:

(i) $\Rightarrow$ (ii): Let $A^{(k)}D + D(A^{(k)})^T \succ 0$ for all k. Then

$$(A^{(k)} \circ S^{(k)})D + D(A^{(k)} \circ S^{(k)})^T = (A^{(k)}D + DA^{(k)}) \circ S^{(k)} \succ 0. \qquad (7)$$

(ii) $\Rightarrow$ (iii): $\mathcal{P}$-set property is a necessary condition of simultaneous diagonal stability.

(iii) $\Rightarrow$ (i): Any $X^{(k)} \succeq 0$ can be expressed in the form $X^{(k)} = D^{(k)}S^{(k)}D^{(k)}$ for some $S^{(k)} \succeq 0$, whose diagonal entries all equal to 1, where $D^{(k)}$ is the diagonal matrix with $D^{(k)}_{i,i} = \sqrt{X^{(k)}_{i,i}}$, $i = 1, 2, \dots, n$. Let $y^{(k)} \in \mathbb{R}^n$ be such that $y^{(k)}_i = D^{(k)}_{i,i}$ for all i. Then,

$$\left[\sum_{k=1}^{m} (A^{(k)} \circ S^{(k)}) y^{(k)} (y^{(k)})^T\right]_{j,j} = \left[\sum_{k=1}^{m} A^{(k)} X^{(k)}\right]_{j,j} \qquad (8)$$

Theorem

Assume $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$. Then, $\mathcal{A}$ is a $\mathcal{P}$-set if and only if $\sum_{k=1}^{m} A^{(k)} \circ y^{(k)}(y^{(k)})^T$ is a P-matrix for any $y^{(k)} \in \mathbb{R}^n$, $k = 1, 2, \ldots, m$, such that for each index i, $y_i^{(k)} \neq 0$ for some $1 \leq k \leq m$.

Theorem

Assume $\mathcal{A} = \{A^{(1)}, A^{(2)}, \dots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$. Then, $\mathcal{A}$ is a $\mathcal{P}$-set if and only if $\sum_{k=1}^{m} A^{(k)} \circ y^{(k)}(y^{(k)})^T$ is a P-matrix for any $y^{(k)} \in \mathbb{R}^n$, $k = 1, 2, \dots, m$, such that for each index i, $y_i^{(k)} \neq 0$ for some $1 \leq k \leq m$.

Main Theorem-2

Given $\mathcal{A} = \{A^{(1)}, A^{(2)}, \dots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$, the following are equivalent:

(i) $\mathcal{A}$ has a common diagonal solution.

(ii) $\sum_{k=1}^{m} A^{(k)} \circ S^{(k)}$ is a P-matrix for all $S^{(k)} \succeq 0$, $k = 1, 2, \dots, m$, provided that for any index $1 \leq i \leq n$, $S_{i,i}^{(k)} = 1$ for some $1 \leq k \leq m$.

(iii) $A_{i,i}^{(k)} > 0$ for $i = 1, 2, \dots, n$ and $k = 1, 2, \dots, m$, and $\det\left(\sum_{k=1}^{m} A^{(k)} \circ S^{(k)}\right) > 0$ for all $S^{(k)} \succeq 0$, $k = 1, 2, \dots, m$, provided that for any index $1 \leq i \leq n$, $S_{i,i}^{(k)} = 1$ for some $1 \leq k \leq m$.

α-Stability

- Consider a partition $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ of $\langle n \rangle$, where $\langle n \rangle = \alpha_1 \cup \cdots \cup \alpha_p$ with these α_k being nonempty and mutually exclusive. When $p = 1$, we simply write $\alpha = \langle n \rangle$.
- A block diagonal matrix with diagonal blocks indexed by $\alpha_1, \ldots, \alpha_p$ is said to be α-diagonal.
- A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is called α-scalar if, for each $1 \leq k \leq p$, $D[\alpha_k]$ is a scalar multiple of the identity matrix of the same size.

α-diagonal

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix}$$

$A_k \in \mathbb{R}^{n_k \times n_k}$ for $n_k = |\alpha_k|$

α-scalar

$$D = \begin{bmatrix} c_1 I_1 & & & \\ & c_2 I_2 & & \\ & & \ddots & \\ & & & c_p I_p \end{bmatrix}$$

$I_k \in \mathbb{R}^{n_k \times n_k}$ for $n_k = |\alpha_k|$

Definition (Hershkowitz and Mashal, 1998)

Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a of $\langle n \rangle$. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be $H(\alpha)$-stable (-semistable) if AH is stable (semistable) for any positive definite α-diagonal matrix H.

- In particular, $H(\langle n \rangle)$-stability is also called H-stability.

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- In particular, $H(\langle n \rangle)$-stability is also called H-stability.

Definition (Hershkowitz and Mashal, 1998)

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be Lyapunov α-scalar stable (semistable) if there exists some positive definite α-scalar matrix D such that

$$AD + DA^T \succ 0 \;\; (AD + DA^T \succeq 0).$$

- We shall abbreviate Lyapunov α-scalar stability as $L(\alpha)$-stability and use the term L-stability when $\alpha = \langle n \rangle$.

Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be additive D-stable (-semistable) if $A + D$ is stable (semistable) for any nonnegative diagonal matrix D.

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- Additive D-stability arises in diffusion models of biological systems after linearization at the equilibrium, and guarantees the asymptotic stability of the equilibrium.
- Additive D-stability has also found applications in neural networks, mathematical economics and mathematical ecology.

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Theorem

Let $A \in \mathbb{R}^{n\times n}$. Then, A is additive D-stable if A is stable and $L(\alpha)$-semistable for some partition α of $\langle n\rangle$,

Definition

Let α be a partition of $\langle n \rangle$. Then, a matrix $A \in \mathbb{R}^{n \times n}$ is said to be additive $H(\alpha)$-stable (-semistable) if $A + H$ is stable (semistable) for any positive semidefinite α-diagonal matrix H.

- When $\alpha = \{\{1\}, \ldots, \{n\}\}$, additive $H(\alpha)$-stability is same as additive D-stability. When $\alpha = \langle n \rangle$, we also use the term additive H-stability in place of $H(\langle n \rangle)$-stability.

Definition

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- When $\alpha = \{\{1\}, \ldots, \{n\}\}$, additive $H(\alpha)$-stability is same as additive D-stability. When $\alpha = \langle n \rangle$, we also use the term additive H-stability in place of $H(\langle n \rangle)$-stability.
- Additive $H(\alpha)$-stability can be interpreted as a criterion for the equilibrium of the following general diffusion problem to be asymptotically stable:

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} h_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + f(u),$$

where $H = [h_{i,j}] \succeq 0$. Additive $H(\alpha)$-stability arises if, in addition, H has an α-diagonal structure.

Lemma (Fiedler and Ptak, 1966)

Let $A \in \mathbb{R}^{n \times n}$. Then, A is a P_0-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, there exists an index i such that $x_i \neq 0$ and $x_i(Ax)_i \geq 0$.

Lemma (Fiedler and Ptak, 1966)

Let $A \in \mathbb{R}^{n\times n}$. Then, A is a P_0-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, there exists an index i such that $x_i \neq 0$ and $x_i(Ax)_i \geq 0$.

Definition

Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a partition of $\langle n \rangle$. A nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is said to be a $P_0(\alpha)$-matrix if for any nonzero $x \in \mathbb{R}^n$, there exists some $1 \leq k \leq p$ such that $(Ax)[\alpha_k] \neq 0$ and $x[\alpha_k]^T(Ax)[\alpha_k] \geq 0$.

- For given $\beta \subseteq \langle n \rangle$, $x[\beta]$ is the subvector of x indexed by β.

Lemma (Fiedler and Ptak, 1966)

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- For given $\beta \subseteq \langle n\rangle$, $x[\beta]$ is the subvector of x indexed by β.
- When $\alpha = \{\{1\}, \ldots, \{n\}\}$, a $P_0(\alpha)$-matrix is a nonsingular P_0-matrix. When $\alpha = \langle n\rangle$, a $P_0(\alpha)$-matrix is a nonsingular positive semidefinite, but not necessarily symmetric, matrix.
- The notion of $P_0(\alpha)$-matrices bridges such general positive semidefinite matrices and nonsingular P_0-matrices.

Main Results

Regular matrix stability

A is diagonally stable.

$\Downarrow$

A is additive D-stable.

$\Downarrow$

A is nonsingular P_0-matrix.

$\Updownarrow$

$A + D$ is nonsingular for any nonnegative diagonal matrix D

α-stability

A is $L(\alpha)$-stable.

$\Downarrow$

A is additive $H(\alpha)$-stable.

$\Downarrow$

A is a $P_0(\alpha)$-matrix.

$\Updownarrow$

$A + H$ is nonsingular for any α-diagonal $H \succeq 0$.

- A one way implication means that the converse does not hold in general.

Main Results

A is H-stable.

$\Downarrow$

A is additive H-stable.

$\Updownarrow$

A is stable and $A + bb^T$ is nonsingular for any $b \in \mathbb{R}^n$.

$\Updownarrow$

A is stable and $A + A^T \succeq 0$.

$\Updownarrow$

A is stable and a $P_0(\langle n \rangle)$-matrix.

A is H-stable.

$\Updownarrow$

$A + P$ is H-stable for any $P \succeq 0$.

A is H-stable.

$\Downarrow$

$A + K$ is L-stable for any $K \succ 0$.

- A one way implication means that the converse does not hold in general.

- $A \in \mathbb{R}^{n \times n}$ is a nonsingular P_0-matrix if and only if $A + D$ is nonsingular for any nonnegative diagonal matrix D if and only if A is nonsingular and $A + D$ is nonsingular for any positive diagonal matrix D.

Conjecture 1

Let α be a partition of $\langle n \rangle$ and $A \in \mathbb{R}^{n \times n}$. Then, the following are equivalent:

(i) A is a $P_0(\alpha)$-matrix.

(ii) $A + H$ is nonsingular for every positive semidefinite α-diagonal matrix H.

(iii) A is nonsingular and $A + H$ is nonsingular for every positive definite α-diagonal matrix H.

Conjecture 2

Let α be a partition of $\langle n \rangle$ and let $A \in \mathbb{R}^{n \times n}$. If A is $H(\alpha)$-stable, then A is a $P_0(\alpha)$-matrix.

On-going work

Theorem (Hershkowitz and Mashal, 1998)

Let $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then, the following statements are equivalent for a matrix A:

(i) A is $L(\alpha)$-stable.

(ii) For every nonzero $X \succeq 0$, there exists some $1 \le k \le r$ such that $\mathrm{tr}\big((AX)[\alpha_k]\big) > 0$.

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(i) A is $L(\alpha)$-stable.

(ii) $A \circ S$ is a $P(\alpha)$-matrix for all $S \succeq 0$ with diagonal entries all being 1.

- $A \in \mathbb{R}^{n \times n}$ is said to be a $P(\alpha)$-matrix if for any nonzero $x \in \mathbb{R}^n$, there exists some $1 \le k \le r$ such that $x[\alpha_k]^T (Ax)[\alpha_k] > 0$.

Definition

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$ and α be a partition of $\langle n\rangle$. If there exists some positive definite α-scalar matrix D such that

$$DA^{(j)} + (A^{(j)})^T D \succ 0, \; j = 1, 2, \ldots, m, \tag{9}$$

then D is called a common $L(\alpha)$-solution for the matrix set $\mathcal{A}$. The existence of such a D is interpreted as the simultaneous $L(\alpha)$-stability of all the matrices in $\mathcal{A}$.

Definition

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$ and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n\rangle$. Then we define $\mathcal{A}$ as a $P(\alpha)$-set if for any vector $x^{(j)} \in \mathbb{R}^n$, $j = 1, 2, \ldots, m$, not all of them zero, there exists $1 \leq k \leq r$ such that

$$\sum_{j=1}^{m} x^{(j)}[\alpha_k]^T \left(A^{(j)} x^{(j)}\right)[\alpha_k] > 0.$$

.

Theorem

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$ and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n\rangle$. Then, $\mathcal{A}$ has a common $L(\alpha)$-solution if and only if for any $X^{(j)} \succeq 0$, $j = 1, \ldots, m$, not all of them zero, there exist $1 \leq k \leq r$ such that

$$\mathrm{tr}\Big(\sum_{j=1}^{m} \big(A^{(j)}X^{(j)}\big)[\alpha_k]\Big) > 0.$$

Theorem

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$ and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then, $\mathcal{A}$ has a common $L(\alpha)$-solution if and only if for any $X^{(j)} \succeq 0$, $j = 1, \ldots, m$, not all of them zero, there exist $1 \leq k \leq r$ such that

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Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n\times n}$ and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then, the following are equivalent:

(i) $\mathcal{A}$ has a common $L(\alpha)$-solution.

(ii) $\{A^{(1)} \circ S^{(1)}, A^{(2)} \circ S^{(2)}, \ldots, A^{(m)} \circ S^{(m)}\}$ is a $P(\alpha)$-set for all $S^{(j)} \succeq 0$, $j = 1, 2, \ldots, m$, with all diagonal entries are equal to 1.

(iii) $\sum_{j=1}^{m} A^{(j)} \circ S^{(j)}$ is a $P(\alpha)$-matrix for all $S^{(j)} \succeq 0$, $j = 1, 2, \ldots, m$, provided that for any index $1 \leq i \leq n$, $S^{(j)}_{i,i} = 1$ for some $1 \leq j \leq m$.

Future works

- Explicit algebraic conditions for the diagonal stability and the simultaneous diagonal stability of higher order matrices.
- Extension of simultaneous diagonal stability problem to the simultaneous $L(\alpha)$-stability case.
- Characterization of $H(\alpha)$-stability and additive $H(\alpha)$-stability.
- Stability properties of structured matrices.

Future works

- Sinc matrix $I^{(-1)} = S + \frac{1}{2}ee^T$, where $e \in \mathbb{R}^n$ is the vector of all ones and

$$S = \begin{bmatrix} s_0 & -s_1 & -s_2 & \cdots & -s_{n-1} \\ s_1 & s_0 & -s_1 & \cdots & -s_{n-2} \\ s_2 & s_1 & s_0 & \cdots & -s_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_0 \end{bmatrix},$$

and $s_k = \int_0^k sinc(x)dx$, where $sinc(x) = \frac{sin(\pi x)}{\pi x}$, $\forall x \neq 0$, while $sinc(0) = 1$.

- S is a skew-symmetric and Toeplitz matrix.
- A recent result confirmed that the Sinc matrix $I^{(-1)}$ is stable, but it is still unknown yet as to whether this matrix has D-stability, a problem key to various applications of Sinc methods.

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THANK YOU