Integral Quadratic Forms and Lattices Satisfying Regularity Conditions

Andrew G. Earnest
Department of Mathematics
Southern Illinois University Carbondale

SIUC Department of Mathematics Colloquium
October 6, 2016

QUADRATIC LATTICES

Let $\mathcal{O}$ be an integral domain with quotient field F. A quadratic $\mathcal{O}$-lattice is a finitely generated $\mathcal{O}$-module L equipped with a symmetric bilinear form $B : L \times L \to F$ and the corresponding quadratic mapping $Q(v) := B(v, v)$ for $v \in L$.

The lattice L is said to be integral if $Q(v) \in \mathcal{O}$ for all $v \in L$, and an integral lattice L is said to be primitive if the ideal of $\mathcal{O}$ generated by $Q(L) = \{Q(v) : v \in L\}$ equals $\mathcal{O}$.

Let K and L be quadratic $\mathcal{O}$-lattices. Then L represents K, denoted $K \to L$, if there exists an injective $\mathcal{O}$-homomorphism $\varphi : K \to L$ such that

$$B_L(\varphi x, \varphi y) = B_K(x, y) \quad \text{for all} \quad x, y \in K.$$

If φ is surjective, then φ is an isometry and K and L are said to be isometric, denoted $K \cong L$. A representation $\varphi : K \to L$ is primitive if $\varphi(K)$ is a direct summand of L; we denote this by $K \xrightarrow{*} L$.

A CLASSICAL EXAMPLE - SUMS OF SQUARES

Consider the free $\mathbb{Z}$-lattice $L = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$. A typical vector $v \in L$ can be written as $v = \sum_{i=1}^n x_i v_i$ for some $x_i \in \mathbb{Z}$. Then

$$Q(v) = \sum_{1 \le i,j \le n} B(v_i, v_j) x_i x_j.$$

For example, the lattice I_n for which $B(v_i, v_j) = \delta_{ij}$ corresponds to the sum of n squares. For a positive integer a, let K_a denote a rank one lattice $\mathbb{Z}y$ equipped with the quadratic mapping $Q(y) = a$. Then $K_a \to I_n$ holds if and only if a can be expressed as a sum of n squares of integers.

Lagrange's Four-Square Theorem (1770): Every positive integer can be expressed as a sum of four squares of integers.

UNIVERSAL LATTICES

For the remainder of the talk, we will consider the case when $\mathcal{O}$ is the ring of integers of a totally real algebraic number field F.

A quadratic $\mathcal{O}$-lattice L is positive definite if $Q(v)$ is a totally positive element of F for all $0 \neq v \in L$.

Definition: For a positive integer k, a positive definite integral quadratic $\mathcal{O}$-lattice is k-universal if it represents every positive definite integral $\mathcal{O}$-lattice of rank k.

Let $\mathcal{U}_{n,k}$ denote the set of isometry classes of k-universal positive definite integral quadratic $\mathbb{Z}$-lattices of rank n.

Classical examples:

- $I_4 \in \mathcal{U}_{4,1}$ (Lagrange 1770)
- $I_5 \in \mathcal{U}_{5,2}$ (Mordell 1930)

THE SETS $\mathcal{U}_{n,k}$

Finiteness/Infiniteness:

- $\mathcal{U}_{n,1} = \emptyset$ for $n \leq 3$
- $\mathcal{U}_{4,1} \neq \emptyset \implies |\mathcal{U}_{n,1}| = \infty \text{ for all } n \geq 5$
- $|\mathcal{U}_{4,1}| < \infty$ (Ross 1946)

THE SETS $\mathcal{U}_{n,k}$

Finiteness/Infiniteness:

- $\mathcal{U}_{n,1} = \emptyset$ for $n \leq 3$
- $\mathcal{U}_{4,1} \neq \emptyset \implies |\mathcal{U}_{n,1}| = \infty \text{ for all } n \geq 5$
- $|\mathcal{U}_{4,1}| < \infty$ (Ross 1946)

The systematic search for all universal rank 4 lattices, which has a long history including a paper of Ramanujan (1917), has recently been completed using the Bhargava-Hanke 290-Theorem.

THE SETS $\mathcal{U}_{n,k}$

Finiteness/Infiniteness:

- $\mathcal{U}_{n,1} = \emptyset$ for $n \leq 3$
- $\mathcal{U}_{4,1} \neq \emptyset \implies |\mathcal{U}_{n,1}| = \infty \text{ for all } n \geq 5$
- $|\mathcal{U}_{4,1}| < \infty$ (Ross 1946)

The systematic search for all universal rank 4 lattices, which has a long history including a paper of Ramanujan (1917), has recently been completed using the Bhargava-Hanke 290-Theorem.

For $k \in \mathbb{N}$, let $u(k)$ denote the minimal n for which $\mathcal{U}_{n,k}$ is nonempty.

- $u(k) = k + 3 \text{ for } 1 \leq k \leq 5$

THE SETS $\mathcal{U}_{n,k}$

Finiteness/Infiniteness:

- $\mathcal{U}_{n,1} = \emptyset$ for $n \leq 3$
- $\mathcal{U}_{4,1} \neq \emptyset \implies |\mathcal{U}_{n,1}| = \infty$ for all $n \geq 5$
- $|\mathcal{U}_{4,1}| < \infty$ (Ross 1946)

The systematic search for all universal rank 4 lattices, which has a long history including a paper of Ramanujan (1917), has recently been completed using the Bhargava-Hanke 290-Theorem.

For $k \in \mathbb{N}$, let $u(k)$ denote the minimal n for which $\mathcal{U}_{n,k}$ is nonempty.

- $u(k) = k + 3$ for $1 \leq k \leq 5$
- $u(6) = 13$,

THE SETS $\mathcal{U}_{n,k}$

Finiteness/Infiniteness:

- $\mathcal{U}_{n,1} = \emptyset$ for $n \leq 3$
- $\mathcal{U}_{4,1} \neq \emptyset \implies |\mathcal{U}_{n,1}| = \infty \text{ for all } n \geq 5$
- $|\mathcal{U}_{4,1}| < \infty$ (Ross 1946)

The systematic search for all universal rank 4 lattices, which has a long history including a paper of Ramanujan (1917), has recently been completed using the Bhargava-Hanke 290-Theorem.

For $k \in \mathbb{N}$, let $u(k)$ denote the minimal n for which $\mathcal{U}_{n,k}$ is nonempty.

- $u(k) = k + 3 \text{ for } 1 \leq k \leq 5$
- $u(6) = 13$, $u(10) = 30$,

THE SETS $\mathcal{U}_{n,k}$

Finiteness/Infiniteness:

- $\mathcal{U}_{n,1} = \emptyset$ for $n \leq 3$
- $\mathcal{U}_{4,1} \neq \emptyset \implies |\mathcal{U}_{n,1}| = \infty$ for all $n \geq 5$
- $|\mathcal{U}_{4,1}| < \infty$ (Ross 1946)

The systematic search for all universal rank 4 lattices, which has a long history including a paper of Ramanujan (1917), has recently been completed using the Bhargava-Hanke 290-Theorem.

For $k \in \mathbb{N}$, let $u(k)$ denote the minimal n for which $\mathcal{U}_{n,k}$ is nonempty.

- $u(k) = k + 3$ for $1 \leq k \leq 5$
- $u(6) = 13$, $u(10) = 30$, $u(24) \geq 6673$ (Oh 1999)

SUMS OF SQUARES REVISITED

The lattice I_3 is not universal. For example, it is easy to see that 7 cannot be written as a sum of three squares of integers. Moreover, it can be seen that the congruence

$$x_1^2 + x_2^2 + x_3^3 \equiv 4^k(8\ell + 7) \pmod{2^{2k+3}}$$

is not solvable in integers for any positive integers k, ℓ.

SUMS OF SQUARES REVISITED

The lattice I_3 is not universal. For example, it is easy to see that 7 cannot be written as a sum of three squares of integers. Moreover, it can be seen that the congruence

$$x_1^2 + x_2^2 + x_3^3 \equiv 4^k(8\ell + 7) \pmod{2^{2k+3}}$$

is not solvable in integers for any positive integers k, ℓ.

Legendre's Three-Square Theorem (1798): Every positive integer not of the type $4^k(8\ell + 7)$ can be expressed as a sum of three squares of integers.

Thus, the sum of three squares is an example of a positive definite integral quadratic form that is "regular" in the sense introduced by L.E. Dickson in 1927; that is, it represents all positive integers not excluded for representation by congruence conditions.

LOCALIZATION

Let $\mathbb{Q}$ denote the field of rational numbers and, for a prime p, let $\mathbb{Q}_p$ denote the field of p-adic numbers (that is, the completion of $\mathbb{Q}$ with respect to the p-adic metric) and $\mathbb{Z}_p$ the ring of p-adic integers (that is, the integral closure of $\mathbb{Z}$ in $\mathbb{Q}_p$).

A quadratic $\mathbb{Z}$-lattice L can be extended to a quadratic $\mathbb{Z}_p$-lattice L_p by extension of scalars (more formally, $L_p = L \bigotimes_{\mathbb{Z}} \mathbb{Z}_p$) and extending the bilinear form on L to L_p by linearity.

As any representation from a quadratic $\mathbb{Z}$-lattice K to a quadratic $\mathbb{Z}$-lattice L can be uniquely extended to a representation from K_p to L_p, we have:

$$K \underset{\mathbb{Z}}{\to} L \quad \Longrightarrow \quad K_p \underset{\mathbb{Z}_p}{\to} L_p \quad \forall\, p.$$

Here "$\forall\, p$" is used as an abbreviation for "for all primes p".

THE GENUS OF A LATTICE AND k-REGULARITY

For a quadratic $\mathbb{Z}$-lattice L, the genus of L is defined to be:

$$\mathrm{gen}\, L = \{K : K_p \cong L_p \,\forall\, p\}.$$

Two basic facts:

- The genus of L contains only a finite number of isometry classes, which is called the class number of L, denoted $h(L)$.
- If $K_p \underset{\mathbb{Z}_p}{\longrightarrow} L_p \,\forall\, p$, then there exists $L' \in \mathrm{gen}\, L$ such that $K \underset{\mathbb{Z}}{\longrightarrow} L'$.

THE GENUS OF A LATTICE AND k-REGULARITY

For a quadratic $\mathbb{Z}$-lattice L, the genus of L is defined to be:

$$\operatorname{gen} L = \{K : K_p \cong L_p \,\forall\, p\}.$$

Two basic facts:

- The genus of L contains only a finite number of isometry classes, which is called the class number of L, denoted $h(L)$.
- If $K_p \underset{\mathbb{Z}_p}{\rightarrow} L_p \,\forall\, p$, then there exists $L' \in \operatorname{gen} L$ such that $K \underset{\mathbb{Z}}{\rightarrow} L'$.

Definition: A positive definite quadratic $\mathbb{Z}$-lattice L of rank n is k-regular for some positive integer $k \leq n$ if for all positive definite quadratic $\mathbb{Z}$-lattices K of rank k,

$$K_p \underset{\mathbb{Z}_p}{\rightarrow} L_p \quad \forall\, p \quad \Longrightarrow \quad K \underset{\mathbb{Z}}{\rightarrow} L.$$

A HIERARCHY OF REGULARITY CONDITIONS

For $n, k \in \mathbb{N}$ with $k \leq n$, let $\mathcal{R}_{n,k}$ be the set of isometry classes of k-regular positive definite primitive integral quadratic $\mathbb{Z}$-lattices of rank n. Note that $\mathcal{U}_{n,k} \subseteq \mathcal{R}_{n,k}$.

A HIERARCHY OF REGULARITY CONDITIONS

For $n, k \in \mathbb{N}$ with $k \leq n$, let $\mathcal{R}_{n,k}$ be the set of isometry classes of k-regular positive definite primitive integral quadratic $\mathbb{Z}$-lattices of rank n. Note that $\mathcal{U}_{n,k} \subseteq \mathcal{R}_{n,k}$.

Proposition: $\mathcal{R}_{n,k} \subseteq \mathcal{R}_{n,m}$ for all $m \leq k$.

Proof: Let $L \in \mathcal{R}_{n,k}$ and suppose $\operatorname{rank} M = m$ and $M_p \to L_p \, \forall \, p$. Then $\exists \, L' \in \operatorname{gen} L$ and a repn. $\sigma : M \to L'$. Since $L' \in \operatorname{gen} L$, for each p there exists an isometry $\Sigma(p) : L'_p \to L_p$. Let K be any sublattice of L' of rank k which contains $\sigma(M)$. Then $K_p \to L_p \, \forall \, p$, since $\Sigma(p) K_p \subseteq L_p$. Since $L \in \mathcal{R}_{n,k}$, it follows that there exists a repn. $\tau : K \to L$. Then $(\tau\sigma)(M) \subseteq L$ and therefore $M \to L$. We conclude that $L \in \mathcal{R}_{n,m}$.

A HIERARCHY OF REGULARITY CONDITIONS

For $n, k \in \mathbb{N}$ with $k \leq n$, let $\mathcal{R}_{n,k}$ be the set of isometry classes of k-regular positive definite primitive integral quadratic $\mathbb{Z}$-lattices of rank n. Note that $\mathcal{U}_{n,k} \subseteq \mathcal{R}_{n,k}$.

Proposition: $\mathcal{R}_{n,k} \subseteq \mathcal{R}_{n,m}$ for all $m \leq k$.

Proof: Let $L \in \mathcal{R}_{n,k}$ and suppose rank $M = m$ and $M_p \to L_p \,\forall\, p$. Then $\exists\, L' \in \operatorname{gen} L$ and a repn. $\sigma : M \to L'$. Since $L' \in \operatorname{gen} L$, for each p there exists an isometry $\Sigma(p) : L'_p \to L_p$. Let K be any sublattice of L' of rank k which contains $\sigma(M)$. Then $K_p \to L_p \,\forall\, p$, since $\Sigma(p)K_p \subseteq L_p$. Since $L \in \mathcal{R}_{n,k}$, it follows that there exists a repn. $\tau : K \to L$. Then $(\tau\sigma)(M) \subseteq L$ and therefore $M \to L$. We conclude that $L \in \mathcal{R}_{n,m}$.

So for each positive integer n, we have:

$$\boxed{\mathcal{R}_{n,n} \subseteq \mathcal{R}_{n,n-1} \subseteq \mathcal{R}_{n,n-2} \subseteq \cdots \subseteq \mathcal{R}_{n,2} \subseteq \mathcal{R}_{n,1}}$$

LATTICES IN $\mathcal{R}_{n,n}$

Restatement: $L \in \mathcal{R}_{n,n} \iff h(L) = 1$

Finiteness:

- $I_n \in \mathcal{R}_{n,n} \iff 1 \leq n \leq 8$
- $|\mathcal{R}_{n,n}| < \infty$ for all n
- $\mathcal{R}_{n,n} \neq \emptyset \iff 1 \leq n \leq 10$ (Watson 1963)

LATTICES IN $\mathcal{R}_{n,n}$

Restatement: $L \in \mathcal{R}_{n,n} \iff h(L) = 1$

Finiteness:

- $I_n \in \mathcal{R}_{n,n} \iff 1 \leq n \leq 8$
- $|\mathcal{R}_{n,n}| < \infty$ for all n
- $\mathcal{R}_{n,n} \neq \emptyset \iff 1 \leq n \leq 10$ (Watson 1963)

Enumeration:

The problem of determining all positive definite lattices with class number one was undertaken by Watson and completed for $n \geq 6$ in a series of papers from 1963 through 1985, and was completed for $3 \leq n \leq 5$ by Kirschmer and Lorch 2013.

LATTICES IN $\mathcal{R}_{n,n-1}$

Theorem (Kitaoka 1978): Let L and L' be nonisometric positive definite quadratic $\mathbb{Z}$-lattices of rank $n \geq 2$ in the same genus. Then there exists a lattice K of rank $(n-1)$ such that $K \to L$ but $K \not\to L'$.

To paraphrase: positive definite $\mathbb{Z}$-lattices are determined up to isometry by local data and the lattices of codimension one that they represent.

In our notation, this theorem implies that

$$\mathcal{R}_{n,n-1} = \mathcal{R}_{n,n}.$$

Remark: This theorem was generalized by Nick Meyer in his thesis in 2015 to positive definite $\mathcal{O}$-lattices over the ring of integers $\mathcal{O}$ of any totally real number field.

LATTICES IN $\mathcal{R}_{n,1}$

Finiteness/Infiniteness:

- $|\mathcal{R}_{3,1}| < \infty$ (Watson 1953)
- $|\mathcal{U}_{n.1}| = \infty \implies |\mathcal{R}_{n,1}| = \infty$ for all $n \geq 5$

LATTICES IN $\mathcal{R}_{n,1}$

Finiteness/Infiniteness:

- $|\mathcal{R}_{3,1}| < \infty$ (Watson 1953)
- $|\mathcal{U}_{n.1}| = \infty \implies |\mathcal{R}_{n,1}| = \infty$ for all $n \geq 5$
- $|\mathcal{R}_{4,1}| = \infty$ (E 1995)

LATTICES IN $\mathcal{R}_{n,1}$

Finiteness/Infiniteness:

- $|\mathcal{R}_{3,1}| < \infty$ (Watson 1953)
- $|\mathcal{U}_{n.1}| = \infty \implies |\mathcal{R}_{n,1}| = \infty$ for all $n \geq 5$
- $|\mathcal{R}_{4,1}| = \infty$ (E 1995)

Enumeration of lattices in $\mathcal{R}_{3,1}$:

- There are 102 diagonal lattices in $\mathcal{R}_{3,1}$ (Jones 1928)
- There are at most 913 lattices in $\mathcal{R}_{3,1}$ (Watson 1953; Jagy, Kaplansky & Schiemann 1997). Among them, 22 were originally identified only as "candidates"; of these, 8 more have now been proven to be regular (Oh 2011) and the remaining 14 have been conditionally proven to be regular under a suitable Generalized Riemann Hypothesis (Lemke Oliver 2014).

k-REGULAR $\mathbb{Z}$-LATTICES, $k \geq 2$

Finiteness/Infiniteness

- $|\mathcal{R}_{4,2}| < \infty$ (E 1994)
- $|\mathcal{U}_{n,2}| = \infty \implies |\mathcal{R}_{n,2}| = \infty$ for all $n \geq 6$

k-REGULAR $\mathbb{Z}$-LATTICES, $k \geq 2$

Finiteness/Infiniteness

- $|\mathcal{R}_{4,2}| < \infty$ (E 1994)
- $|\mathcal{U}_{n,2}| = \infty \implies |\mathcal{R}_{n,2}| = \infty$ for all $n \geq 6$
- $|\mathcal{R}_{5,2}| < \infty$ (Chung 2001)
- $|\mathcal{R}_{n,n-3}| < \infty$ for all $n \geq 5$ (Chan & Oh 2003)

k-REGULAR $\mathbb{Z}$-LATTICES, $k \geq 2$

Finiteness/Infiniteness

- $|\mathcal{R}_{4,2}| < \infty$ (E 1994)
- $|\mathcal{U}_{n,2}| = \infty \implies |\mathcal{R}_{n,2}| = \infty$ for all $n \geq 6$
- $|\mathcal{R}_{5,2}| < \infty$ (Chung 2001)
- $|\mathcal{R}_{n,n-3}| < \infty$ for all $n \geq 5$ (Chan & Oh 2003)

Enumeration

- There are 177 even lattices in $\mathcal{R}_{4,2}$, all of which have class number 1. (Oh 2008)
- Do there exist any $L \in \mathcal{R}_{4,2}$ with $h(L) > 1$? That is, is $\mathcal{R}_{4,2} = \mathcal{R}_{4,4}$?

Open Question: Is there any $n \geq 4$ for which $\mathcal{R}_{n,n-2} \neq \mathcal{R}_{n,n}$?

SOME EXAMPLES

$n = 6$:

$$\mathcal{R}_{6,6} = \mathcal{R}_{6,5} \stackrel{?}{=} \mathcal{R}_{6,4} \subseteq \mathcal{R}_{6,3} \subseteq \mathcal{R}_{6,2} \subseteq \mathcal{R}_{6,1}$$

SOME EXAMPLES

$n = 6$:

$$\mathcal{R}_{6,6} = \mathcal{R}_{6,5} \stackrel{?}{=} \mathcal{R}_{6,4} \subseteq \mathcal{R}_{6,3} \subseteq \mathcal{R}_{6,2} \subseteq \mathcal{R}_{6,1}$$

$n = 10$:

$$\mathcal{R}_{10,10} \subseteq \cdots \subseteq \mathcal{R}_{10,7} \subseteq \mathcal{R}_{10,6} \subseteq \mathcal{R}_{10,5} \subseteq \cdots \mathcal{R}_{10,1}$$

Question: Is $\mathcal{R}_{10,6}$ finite or infinite?

STRICTLY REGULAR $\mathbb{Z}$-LATTICES

Definition: A positive definite quadratic $\mathbb{Z}$-lattice L of rank n is strictly k-regular for some positive integer $k \leq n$ if for all positive definite quadratic $\mathbb{Z}$-lattices K of rank k,

$$K_p \xrightarrow{*} L_p \, \forall \, p \quad \Longrightarrow \quad K \xrightarrow{*} L.$$

For $n, k \in \mathbb{N}$ with $k \leq n$, let $\mathcal{R}^*_{n,k}$ be the set of isometry classes of strictly k-regular positive definite primitive integral quadratic $\mathbb{Z}$-lattices of rank n. It can be shown that $\mathcal{R}^*_{n,k} \subseteq \mathcal{R}_{n,k}$.

Finiteness:

- $|\mathcal{R}^*_{4,1}| < \infty$ (E, Kim & Meyer 2014)
- $|\mathcal{R}^*_{n,n-4}| < \infty$ for all $n \geq 6$ (Marino, in preparation)

Enumeration:

- There are 94 diagonal lattices in $\mathcal{R}^*_{4,1}$ (among these lattices, the largest class number is 7). (E, Kim & Meyer 2015)

AN INTERESTING EXAMPLE AND A CONJECTURE OF KAPLANSKY

Among the 913 regular ternary lattices, there exist three pairs of lattices that are nonisometric but lie in the same genus and are both regular. One such pair consists of lattices L and K corresponding to the quadratic forms

$$x^2 + xy + y^2 + 9z^2 \quad \text{and} \quad x^2 + 3(y^2 + yz + z^2).$$

Consequently, these lattices represent exactly the same integers; that is, $Q(L) = Q(K)$.

AN INTERESTING EXAMPLE AND A CONJECTURE OF KAPLANSKY

Among the 913 regular ternary lattices, there exist three pairs of lattices that are nonisometric but lie in the same genus and are both regular. One such pair consists of lattices L and K corresponding to the quadratic forms

$$x^2 + xy + y^2 + 9z^2 \quad \text{and} \quad x^2 + 3(y^2 + yz + z^2).$$

Consequently, these lattices represent exactly the same integers; that is, $Q(L) = Q(K)$.

Conjecture (Kaplansky): If K and L are nonisometric positive definite ternary $\mathbb{Z}$-lattices that lie in the same genus and have the property that $Q(K) = Q(L)$, then K and L are both regular.

AN INTERESTING EXAMPLE AND A CONJECTURE OF KAPLANSKY

Among the 913 regular ternary lattices, there exist three pairs of lattices that are nonisometric but lie in the same genus and are both regular. One such pair consists of lattices L and K corresponding to the quadratic forms

$$x^2 + xy + y^2 + 9z^2 \quad \text{and} \quad x^2 + 3(y^2 + yz + z^2).$$

Consequently, these lattices represent exactly the same integers; that is, $Q(L) = Q(K)$.

Conjecture (Kaplansky): If K and L are nonisometric positive definite ternary $\mathbb{Z}$-lattices that lie in the same genus and have the property that $Q(K) = Q(L)$, then K and L are both regular.

Open Questions: Are there only finitely many pairs of nonisometric positive definite ternary $\mathbb{Z}$-lattices K and L that lie in the same genus and have the property that $Q(K) = Q(L)$? Any with $Q^*(K) = Q^*(L)$?

ISOSPECTRAL LATTICES

For a positive definite $\mathbb{Z}$-lattice and a positive integer a, let

$$r(L, a) = \# \text{ of representations of a by L.}$$

Theorem (Schiemann 1997): If $\operatorname{rank} L = \operatorname{rank} K = 3$, then

$$r(L, a) = r(K, a) \quad \forall\, a \in \mathbb{N} \quad \Longrightarrow \quad L \cong K.$$

Theorem (Conway & Sloane 1992; Cervino & Hein 2011): There exist infinitely many pairs of nonisometric positive definite $\mathbb{Z}$-lattices K and L of rank 4 which lie in the same genus and have the property that $r(L, a) = r(K, a) \,\forall\, a \in \mathbb{N}$.

FURTHER DIRECTIONS

- spinor regular ternary lattices
- almost regular quadratic lattices
- regular Hermitian lattices over the rings of integers of imaginary quadratic fields
- regular quadratic lattices over $\mathbb{F}_q[T]$
- inhomogeneous quadratic polynomials
- others??

THANK YOU!!!