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Structure and Complexity of Orders on Structures

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- Magma is a nonempty set with a binary operation: $(M, \cdot)$

- A magma is *computable* if its domain is computable and its operation is computable. (Ex: $(\mathbb{Q}, +)$, $(\mathbb{Z}, +)$)

- A *left-order* on magma $(M, \cdot)$ is
a linear ordering $<$ of the domain M, which
it is left-invariant with respect to $\cdot$
$(\forall x, y, z)[x < y \Rightarrow z \cdot x < z \cdot y]$

- $<$ is a *bi-order* (*order*) on the structure if
$(\forall x, y, z)[x < y \Rightarrow (z \cdot x < z \cdot y \wedge x \cdot z < y \cdot z)]$

- Let magma be a group.

- Given a *left order* $\prec_l$ on a group G, we have a *right order* $\prec_r$ on G:
  $x \prec_r y \Leftrightarrow_{def} y^{-1} \prec_l x^{-1}$

  $x \prec_r y \Rightarrow y^{-1} \prec_l x^{-1} \Rightarrow z^{-1}y^{-1} \prec_l z^{-1}x^{-1}$

  $\Rightarrow (yz)^{-1} \prec_l (xz)^{-1} \Rightarrow xz \prec_r yz$

- If a group G is computable, then $\prec_r$ and $\prec_l$ have the same Turing degree.

- G is left-orderable group $\Rightarrow$ G is *torsion-free*

  torsion-free: $(\forall x \in G - \{e\})[order(x) = \infty]$

  $e \prec x \Rightarrow x \prec x^2 \prec x^3 \prec \cdots \prec x^n$

- Every torsion-free abelian group is orderable.

- Every torsion-free nilpotent group is orderable.

- Torsion-free, but not left-orderable group:
  $G = \langle x, y \mid xy^2x^{-1}y^2 = e,\ \ yx^2y^{-1}x^2 = e \rangle$
  $= \langle x, y \mid xy^2 = y^{-2}x,\ \ yx^2 = x^{-2}y \rangle$

- $LO(M)$ the set of left orders on M
  $RO(M)$ the set of right orders on M
  $BiO(M)$ the set of bi-orders on M

- $(\mathbb{Z},+)$ has two orders, both computable.

  $(\mathbb{Z}^2,+)$ has $2^{\aleph_0}$ orders.

  $\mathbb{Z}^\omega = \bigoplus_{i\in\omega}\mathbb{Z}$, the direct sum of ω copies of $\mathbb{Z}$

  $(\mathbb{Z}^\omega,+)$ has $2^{\aleph_0}$ orders.

- (Solomon, 2002)

  (i) A computable torsion-free abelian group of finite rank $n > 1$ has an order in every Turing degree.

  (ii) A computable torsion-free abelian group of infinite rank has an order in every Turing degree $\geq \mathbf{0}'$, the degree of the halting set.

  (iii) A computable torsion-free properly n-step nilpotent group G has an order in every Turing degree $\geq \mathbf{0^{(n)}}$, the degree of the nth iteration of the halting set.

- (Downey and Kurtz, 1986)

  There is a computable torsion-free abelian group G (hence orderable) such that G has no computable order.

- (Dobrica, 1983)

  Every computable torsion-free abelian group is isomorphic to a computable group with a computable basis (hence with a computable order).

- (Harrison-Trainor, 2017)

  There is a computable left-orderable group that is not isomorphic to a computable group with a computable left-order.

  Not known for the case of bi-orderable groups.

- For groups, orders are often identified with their positive cones.

  Let $\prec$ be a partial left order on a group G.

  Positive cone: $P = \{a \in G : a \succeq e\}$
  Negative cone: $P^{-1} = \{a \in G : a \preceq e\}$

  1. $PP \subseteq P$ (P *sub-semigroup* of G)
  2. $P \cap P^{-1} = \{e\}$

  P with 1 & 2 defines a partial left order $\preceq_P$ on G:

$$x \preceq_P y \Leftrightarrow x^{-1}y \in P$$

  $x \preceq_P y \Rightarrow x^{-1}y \in P$

  $\Rightarrow x^{-1}z^{-1}zy = (zx)^{-1}(zy) \in P$

  $\Rightarrow zx \preceq_P zy$

- P with 1 & 2 defines a *left order* iff

  3. $P \cup P^{-1} = G$

- P with 1, 2 & 3 defines a *bi-order* iff:
  4. $(\forall g \in G)[g^{-1}Pg \subseteq P]$ (P *normal*)

  bi-order $\prec$; let $g \in G$
  $y \succ e \Leftrightarrow g^{-1}y \succ g^{-1} \Leftrightarrow g^{-1}yg \succ e$

  P normal; let $x \preceq_P y$, $z \in G$

  right-invariant: $x^{-1}y \in P \Rightarrow z^{-1}x^{-1}yz \in P$
  $(xz)^{-1}yz \in P \Rightarrow xz \preceq_P yz$

- Example: $G = \mathbb{Z} \oplus \mathbb{Z}$ has a bi-order with positive cone

$$P = \{(a,b) \mid a > 0 \vee (a = 0 \wedge b \geq 0)\}.$$

- Fundamental group of Klein bottle
  $K = \left\langle a, b \mid a^{-1}ba = b^{-1} \right\rangle$ left-orderable, but not bi-orderable.
  $ba = ab^{-1}$

  Positive cone $P = \{a^n b^m \mid n > 0 \vee (n = 0 \wedge m \geq 0)\}$
  defines a left order on G.

  Not bi-orderable:
  $b \succ e \Rightarrow a^{-1}ba = b^{-1} \succ e$

- For every bi-orderable (left-orderable) computable magma M, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $BiO(M)$ $(LO(M))$ to the set of all infinite paths of $\mathcal{T}$.

- If a computable magma has only finitely many bi-orders (left orders), they are all computable.

- Hence, by the *Low Basis Theorem* of Jockusch and Soare, $BiO(M)$ $(LO(M))$ contains a low order.
  A set X and its Turing degree $\mathbf{x}$ are called *low* if the degree of the halting set with "oracle" X is the same the degree of the halting set.

- A left-orderable computable magma has a left order of a computably enumerable Turing degree.

- For a bi-order $<$: $(a < b \wedge c < d \Rightarrow a \cdot c < b \cdot d)$
  $a < b \Rightarrow a \cdot c < b \cdot c$
  $c < d \Rightarrow b \cdot c < b \cdot d$

- Not necessarily true for a left order.

  Example: Klein bottle group $K = \left\langle a, b \mid a^{-1}ba = b^{-1} \right\rangle$

  Left-orderable, but not bi-orderable
  $b > e \Leftrightarrow a^{-1}ba = b^{-1} > e$

  $ba = ab^{-1}$

  b and a^2 commute:
  $a^2b = a^2(a^{-1}b^{-1}a) = ab^{-1}a = ba^2$

  $ba \neq a$ but $(ba)^2 = baba = ab^{-1}ba = a^2$

- A magma $(Q, *)$ is called a *quandle* if:

  1. $(\forall a)[a * a = a]$ (idempotence);

  2. for every $b \in Q$, the mapping $*_b : Q \longrightarrow Q$ defined by

  $$*_b(a) = a * b$$

  is bijective;

  3. $(\forall a, b, c)[(a * b) * c = (a * c) * (b * c)]$ (right self-distributivity).

- A quandle Q is called *trivial* if the operation $*$ is defined by

  $$(\forall a, b)[a * b = a].$$

  Every linear ordering of elements of Q is right invariant.

- For a group G, the *conjugate* quandle Conj(G) is one with domain G and the operation $*$ given by $a * b = b^{-1}ab$.

- 1. $a * a = a^{-1}aa = a$

  2. $\forall b\ \forall c\ \exists! a\ [a * b = c]$

  $b^{-1}ab = c \Rightarrow a = bcb^{-1}$

  3. $(a * b) * c = c^{-1}(a * b)c = c^{-1}b^{-1}abc$

  $(a * c) * (b * c) = (c^{-1}ac) * (c^{-1}bc) = (c^{-1}bc)^{-1}(c^{-1}ac)(c^{-1}bc)$
  $= c^{-1}b^{-1}cc^{-1}abc = c^{-1}b^{-1}abc$

  Then every bi-order on G induces a right order on Conj(G).

- Let B be a bi-order on G. Define R on Conj(G) as

$$(\forall a, b)[(a, b) \in R \Leftrightarrow (a, b) \in B]$$

R is right-invariant on Conj(G) because for $(a, b) \in R$ and $c \in$ Conj(G):

$$(a, b) \in B \Rightarrow (c^{-1}ac, c^{-1}bc) \in B \Rightarrow (a * c, b * c) \in R$$

- Not all right orders on Conj(G) are induced by bi-orders on G.
It is possible to have $BiO(G) = \emptyset$,
while $RO(\text{Conj}(G)) \neq \emptyset$.

Let G be an abelian group with torsion.
Then $BiO(G) = \emptyset$, but Conj(G) is a trivial quandle,
so it admits many right orders.

- *Topology* defined on $LO(M)$ by subbasis $\{S_{(a,b)}\}_{(a,b)\in(M\times M)-\Delta}$ where $\Delta = \{(a,a) \mid a \in M\}$:

$$S_{(a,b)} = \{R \in LO(M) \mid (a,b) \in R\}.$$

- (Dabkowska, Dabkowski, Harizanov, Przytycki, and Veve, 2007)

  Let M be a left-orderable magma with cardinality $|M| = \mathfrak{m} \geq \aleph_0$. Then $LO(M)$ is a compact space. $BiO(M)$ is also a compact space.

  By Vedenissoff's theorem, $LO(M)$ can be homeomorphically embedded into the Cantor cube $\{0,1\}^{\mathfrak{m}}$. Moreover, $LO(M)$ is a closed subspace of the Cantor cube $\{0,1\}^{\mathfrak{m}}$.

- If M is a countable magma, then $LO(M)$ is metrizable.

- If $M = G$ is a group, we showed how we could also use the following theorem to establish that $LO(G)$ is compact.

- (Onishi, Łoś, Conrad)
  A partial left order given by its positive cone P can be extended to a total left order on G iff for every $\{x_1, ..., x_n\} \subseteq G \backslash \{e\}$ there are $\epsilon_1, ..., \epsilon_n$, $\epsilon_i \in \{1, -1\}$, such that

$$e \notin sgr((P \backslash \{e\}) \cup \{x_1^{\epsilon_1}, ..., x_n^{\epsilon_n}\}),$$

  where $sgr(A)$ is the sub-semigroup of G generated by A.

- For a countable group G, $LO(G) \neq \emptyset$ is homeomorphic to the Cantor set iff for any sequence $(a_0, b_0), ..., (a_{k-1}, b_{k-1})$, $S_{(a_0,b_0)} \cap \cdots \cap S_{(a_{k-1},b_{k-1})}$ is either empty or infinite.

- (Sikora, 2004)
  The space $LO(\mathbb{Z}^n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Dabkowska, 2006)
  The space $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set.

- There are countable groups with infinitely countably many bi-orders.

- (Linnell, 2006)
  The space of left orders of a countable left-orderable group is either finite or contains a homeomorphic copy of the Cantor set.

- Let $F_n = \langle x_0, x_1, \ldots, x_{n-1} \mid \ \rangle$ be a free group of rank n.

- *Conjecture* (Sikora, 2004)
  For $n > 1$, the spaces $LO(F_n)$ and $BiO(F_n)$ are homeomorphic to the Cantor set.

- (Navas-Flores, 2008)
  The space $LO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

  Not known for $BiO(F_n)$ for $n > 1$.

- (Dabkowska, Dabkowski, Harizanov and Togha, 2010)
  For a computable group G isomorphic to a free group F_n of rank $n > 1$, we have a bi-order in every Turing degree.

- (Chubb, Dabkowski and Harizanov, 2017)
  For a computable group G isomorphic to a free group F_n of rank $n > 1$, we have a bi-order in every strong degree known as truth table degree.

Proof sketch:

For a group G, the *lower central series* is the descending sequence of subgroups $(\gamma_\alpha(G))_\alpha$ defined as:

$$\gamma_1(G) = G,$$
$$\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G],$$
$$\gamma_\beta(G) = \bigcap_{\alpha<\beta} \gamma_\alpha(G), \text{ when } \beta \text{ is a limit ordinal,}$$

where $[A, B]$ is the subgroup of G generated by the elements $a^{-1}b^{-1}ab$, with $a \in A$ and $b \in B$.

- *Lower central series* of F_n: $\gamma_1(F_n) \geq \cdots \geq \gamma_i(F_n) \geq \cdots$

- (Magnus) $\gamma_\omega = \bigcap_{i=1}^{\omega} \gamma_i(F_n) = \{e\}$

- (Hall) $\gamma_i(F_n)/\gamma_{i+1}(F_n) \cong \mathbb{Z}^{k_i}$,
  where $k_i = \frac{1}{i}\sum_{d|i} \mu(\frac{i}{d})n^d$, μ Möbius function

- Isomorphism uniformly computable since a basis of $\gamma_i(F_n)/\gamma_{i+1}(F_n)$ can be found algorithmically in n, i.

- Construct bi-orders on F_n using bi-orders on $\gamma_i(F_n)/\gamma_{i+1}(F_n)$.

- Different choices of orders on quotients induce different orders on F_n.

- Produce a bi-order on F_n of a given Turing degree.

- (Harizanov, Knight, McCoy, Puzarenko, Solomon and Wallbaum, preprint)

  Let $F_\infty = \langle x_0, x_1, \ldots \mid \ \rangle$ be a free group of rank $\aleph_0$.
  There is a computable copy F of F_∞
  with no computable left order (hence no computable bi-order).

- (Ha and Harizanov, to appear)

  Let Conj(F_∞) be the conjugacy quandle of F_∞.
  There is a computable copy M of Conj(F_∞)
  with no computable left order (hence no computable bi-order).

- *Corollary*
  The spaces $LO(F_\infty)$ and$BiO(F_\infty)$ are homeomorphic to the Cantor set.

  The spaces $LO(Conj(F_\infty))$ and $BiO(Conj(F_\infty))$ are homeomorphic to the Cantor set.

- For every left-orderable computable group G, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $LO(G)$ to the set of all infinite paths of $\mathcal{T}$.

- An isolated path in a computable binary tree must be computable.

  In a computable binary tree with infinite paths and no computable ones, the space of paths is homeomorphic to the Cantor set.

  Topology of the space of orders does not change under isomorphisms, so it is the same for the whole isomorphism class.

- We can generalize the construction for free groups of finite rank > 1 to a class of finitely presented, *residually nilpotent* groups that are not nilpotent.

- (Chubb, Dabkowski and Harizanov, 2017)
  Let G be a finitely presented, torsion-free, computable group.
  Let $G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots$ be the lower central series of G.
  If $\gamma_\omega(G) = \{e\}$, and
  $\gamma_i(G)/\gamma_{i+1}(G)$ is nontrivial and torsion-free for each $i = 1, 2, \ldots$,
  then there is a bi-order on G in every Turing degree.

- *Examples*:

  (1) Surface groups of genus $n > 1$:
  $\langle x_1, y_1, \ldots, x_n, y_n \mid [x_1, y_1] \cdots [x_n, y_n] \rangle$

(2) Finitely generated one-relator parafree groups (introduced by Baumslag): residually nilpotent groups and its quotients by the terms of its lower central series are the same as those of a free group.

(3) Right-angled Artin groups A_G, where

G is a graph with vertices $V(G) = \{1, 2, \ldots, n\}$ for $n \geq 2$ and edges $E(G)$,

$A_G = \langle x_1, \ldots, x_n \mid [x_i, x_j], (i, j) \in E(G) \rangle$

THANK YOU!