Closed book and notes. Show all of your work. Please turn in these sheets with your answers. Full credit (200 points) will only be given for rigorous mathematical justification.

\( \lambda \) denotes Lebesgue measure on \( \mathbb{R} \); and \( \lambda^* \) is Lebesgue outer measure.

1. (30 points) Define what is meant by a Lebesgue measurable set in \( \mathbb{R} \) and a Lebesgue measurable function \( f : \mathbb{R} \to \mathbb{R} \).

(a) Let \( f : \mathbb{R} \to [0, \infty) \) be a function such that there are non-negative constants \( c_n, n \geq 1 \) and Lebesgue measurable sets \( E_n, n \geq 1 \), so that

\[
    f(x) = \sum_{n=1}^{\infty} c_n 1_{E_n}(x)
\]

for all \( x \in \mathbb{R} \). The characteristic (indicator) function of the set \( E_n \) is denoted by \( 1_{E_n} \). Prove that \( f \) is Lebesgue measurable. Is \( f \) Borel measurable when each \( E_n \) is Borel measurable?

(b) Let \( f_n : \mathbb{R} \to \mathbb{R}, n \geq 1 \), be a sequence of real-valued Lebesgue measurable functions. Let \( E \) be the set of all \( x \in \mathbb{R} \) such that the sequence \( \{f_n(x)\}_{n=1}^{\infty} \) converges in \( \mathbb{R} \). Prove that \( E \) is Lebesgue measurable.

2. (40 points) Give an outline of the construction of Lebesgue measure \( \lambda \) on \( \mathbb{R} \). Do not give proofs.

(a) Let \( A \subset \mathbb{R} \) be a set of positive Lebesgue measure and suppose that \( 0 < \epsilon < \lambda(A) \). Show that there exists a Lebesgue measurable subset \( B \) of \( A \) such that \( \lambda(B) = \epsilon \).

(b) Let \( E \subset [0, 1] \) be a Lebesgue measurable set such that \( \lambda(E) = 1 \). Show that \( E \) is dense in \([0, 1]\).

(c) Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of subsets of \([0, 1]\) such that \( \sum_{n=1}^{\infty} \lambda^*(A_n) < \infty \), where \( \lambda^* \) is Lebesgue outer measure. Show that the set

\[
    \{ x \in [0, 1] : x \in A_n \text{ for infinitely many } n \}
\]
has Lebesgue measure zero.

3.(30 points) Distinguish between a Lebesgue-measurable set and a Borel measurable set in $\mathbb{R}$. Let $\lambda$ be Lebesgue measure on $\mathbb{R}$. Prove the following statements:

(a) The union of two Lebesgue measurable sets is Lebesgue measurable.

(b) If $E$ is Lebesgue-measurable, then given $\epsilon > 0$, there is an open set $U$ and a closed set $F$ such that $F \subseteq E \subseteq U$ and

$$\lambda(U \setminus E) < \epsilon, \quad \lambda(E \setminus F) < \epsilon$$

(c) For every Lebesgue-measurable set $E$, there is a Borel measurable set $B$ such that $E \subseteq B$ and $\lambda(E) = \lambda(B)$.

(d) For any real-valued function $f : \mathbb{R} \to \mathbb{R}$, the set of points at which $f$ is discontinuous is a Borel set.

4.(30 points) Define what is meant by an extended-real-valued (Lebesgue)-measurable function $f : [a, b] \to \mathbb{R}^*$.

Let $f : [a, b] \to \mathbb{R}^*$ be a measurable function which is finite almost everywhere (with respect to Lebesgue measure), and let $\epsilon > 0$ be given. Prove the following two statements

(a) There exists $K > 0$ such that $|f(x)| \leq K$ for all $x$ outside a set of measure less than $\epsilon$.

(b) There is a simple function $\phi : [a, b] \to \mathbb{R}$ such that $|f(x) - \phi(x)| < \epsilon$ for all $x$ outside a set of measure less than $\epsilon$.

5.(30 points) Define convergence in measure (with respect to Lebesgue measure $\lambda$) for a sequence of measurable extended-real-valued functions $f_n : E \to \mathbb{R}^*$, $n \geq 1$. Prove the following version of Fatou’s Lemma:

Let $E$ be a subset of $\mathbb{R}$ of finite measure. Suppose $f_n : E \to \mathbb{R}^*$, $n \geq 1$, is a sequence of non-negative measurable function such that $f_n \to f$ in measure on $E$. Then

$$\int_E f(x) \, d\lambda(x) \leq \liminf_{n \to \infty} \int f_n(x) \, d\lambda(x).$$
6. (30 points) Define what is meant by a Lebesgue integrable function \( f : (0, \infty) \to \mathbb{R} \).

Consider the function \( f : (0, \infty) \to \mathbb{R} \) defined by
\[
f(x) := \frac{\sin x}{x}
\]
for all \( x > 0 \). Show that the improper Riemann integral \( \int_0^\infty f(x) \, dx \) exists but the Lebesgue integral \( \int_0^\infty f(x) \, d\lambda(x) \) does not exist. \( \text{(Hint: Prove that the series} \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} \, dx \text{is conditionally convergent.)} \)

7. (30 points) State Lebesgue’s Dominated Convergence Theorem and the Monotone Convergence Theorem.

(a) Let \( f_n : \mathbb{R} \to \mathbb{R}^{\geq 0} \) be a sequence of non-negative integrable functions on \( \mathbb{R} \) such that \( f_n \to f \) almost everywhere, and \( f \) is an integrable function over \( \mathbb{R} \) (with respect to Lebesgue measure \( \lambda \)). Suppose that
\[
\lim_{n \to \infty} \int f_n(x) \, d\lambda(x) = \int f(x) \, d\lambda(x).
\]
Show that for any Lebesgue measurable set \( E \), one has
\[
\lim_{n \to \infty} \int_E f_n(x) \, d\lambda(x) = \int_E f(x) \, d\lambda(x)
\]
(b) Show that the following limit exists, evaluate it and justify all your calculations:
\[
\lim_{n \to \infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)} \, d\lambda(x)
\]

8. (30 points) Let \( f : [0, 1] \to \mathbb{R} \) be a Lebesgue integrable function such that
\[
\int_0^1 x^n f(x) \, d\lambda(x) = 0
\]
for all integers \( n \geq 0 \). Show that \( f = 0 \) almost everywhere. Is your conclusion true if the above relation is to hold for all \textit{even} integers:

\[
\int_0^1 x^{2n} f(x) \, d\lambda(x) = 0, \quad n \geq 0?
\]

Give reasons for your answer. (\textit{Hint:} The algebra generated by \( \{1, x^2\} \) contains the constant functions and separates points in \([0, 1]\). Hence it is dense in \( C([0, 1], \mathbb{R}) \) by the Stone-Weierstrass theorem.)

Suppose \( f : [0, 1] \to \mathbb{R} \) is Lebesgue integrable and satisfies

\[
\int_0^1 f(x^{1/(2n+1)}) \, d\lambda(x) = 0
\]

for every integer \( n \geq 0 \). Show that \( f = 0 \) almost everywhere.

9.(30 points) For a sequence of functions \( f_n : [0, \infty) \to \mathbb{R}, n \geq 1 \), define what is meant by \textit{convergence in measure}, \textit{pointwise convergence}, \textit{convergence almost everywhere} and \textit{convergence in} \( L^1 \).

Consider the sequence \( f_n : [0, \infty) \to \mathbb{R}, n \geq 1 \), by

\[
f_n := \sin(1/n)1_{(0,n)}, \quad n \geq 1,
\]

where \( 1_{(0,n)} \) denotes the characteristic (or indicator) function of \((0,n)\). Decide whether the sequence \( \{f_n\}_{n=1}^\infty \) converges pointwise, converges in measure, converges almost everywhere, or converges in \( L^1 \). Justify your answer in each case.

10.(30 points) Let \( f : [0, 1] \to \mathbb{R} \) be absolutely continuous. Prove the following statements:

(a) \( f \) is continuous on \([0, 1]\).

(b) \( f \) is of bounded variation on \([0, 1]\).

(c) If \( E \subset [0, 1] \) has Lebesgue measure zero, then \( f(E) \) also has Lebesgue measure zero.