Ph.D. Qualifying Exam in MATH 530 (Topology)
August 2023

Instructions: Do as many of the ten problems below as you can. Please use a separate sheet of paper for each problem. If a problem takes more than one page, number the pages. Do not use the back of a page. Good luck!

1. (10 points; 2 points each)
   a) What is a Lindelöf space?
   b) Define local compactness.
   c) Define the compactification of a space.
   d) State the Urysohn Lemma.
   e) What is the quotient topology?

2. (10 points) Prove that the continuous image of a compact space is compact. Give an example showing that the continuous image of a closed set need not be closed.

3. (10 points) Construct an example showing that a quotient space of a Hausdorff space need not be Hausdorff.

4. (10 points) Prove the Sequence Lemma: Let $X$ be a topological space. Let $A \subset X$. If there is a sequence of points in $A$ converging to $x$, then $x \in \bar{A}$. The converse holds if $X$ is a metric space.

5. (10 points) Show that $\mathbb{R}^\omega$ in the box topology is not metrizable. Hint: exploit the Sequence Lemma.

6. (10 points) Prove that a topological space $X$ is locally path connected if and only if for every open set $U$ of $X$, each component of $U$ is open in $X$.

7. (10 points) Let $X$ be a compact Hausdorff space. Prove that if $X$ has no isolated points, then $X$ is uncountable.

8. (10 points) Prove that the space $\mathbb{R}_l$ is normal. (Recall $\mathbb{R}_l$ has the topology generated by intervals of the form $[a, b]$.)

9. (10 points) Let $D^2$ be a closed disk. Show that there is no retraction from $D^2$ to its boundary.

10. (10 points) a. What is the fundamental group of $T^2$ with one point deleted? b. What is the fundamental group of $T^2$ with two points deleted? (Hint: draw the torus as a square with opposite sides identified and then delete two points from the interior.)