Ph.D. Qualifying Exam in Topology and Geometry
Spring Semester 2023

Instructions: Do as many of the ten problems below as you can. Please use a separate sheet of paper for each problem. If a problem takes more than one page, number the pages. Do not use the back of a page. Put your name on each page. Good luck!

1. [10 points] (Answer 5 of the 7, for 2 points each.)
   a. Define homeomorphism.
   b. Define second countable.
   c. Define normal.
   d. Define locally compact.
   e. State the Lebesgue Number Lemma.
   f) Define strong deformation retract.
   g) Define covering map.
2. [10 points] In a topological space prove that the intersection of any collection of closed sets is closed.
3. [10 points] Prove the Maximum Value Theorem: Let $f : X \to Y$ be continuous, where X is compact and Y is an ordered set in the order topology. There exists a point $m \in X$ such that $f(x) \leq f(m)$ for every $x \in X$.
4. [10 points] Let X be a regular topological space and let $C \subset X$ be closed. Define an equivalence relation $\sim$ on X by $a \sim b$ if and only if $a = b$ or a and b are both in C. Prove that the quotient space $X/\sim$ in the quotient topology is Hausdorff.
5. [10 points] Prove that every compact Hausdorff space is normal.
6. [10 points] Let X be locally path connected. Suppose that U is an open connected subset of X. Prove that U is path connected. (Hint: think about the case $X = \mathbb{R}^2$.)
7. [10 points] Let A be a closed subset of X, and B be a closed subset of Y. Show that $A \times B$ is a closed subset of $X \times Y$.
8. [10 points] Construct an example showing that a quotient space of a Hausdorff space need not be Hausdorff.
9. [10 points] Prove that every second countable space X is separable, that is X contains a countable dense subset.
10. [10 points] What is the fundamental group of the 2-dimensional sphere with two points deleted? Explain your reasoning.