Math 519 Qualifying Exam

Spring 2023

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let $G$ be a group. Denote by $\text{Aut}(G)$ the group of all automorphisms of $G$ and by $\text{Inn}(G)$ its subgroup consisting of all inner automorphisms (i.e., automorphisms of the form $x \mapsto yxy^{-1}$ for some $y \in G$). Consider a map $j : G \to G$ defined by $j(g) = g^{-1}$. Prove that $|\text{Inn}(G)| = 1$ if and only if $j \in \text{Aut}(G)$.

2. Let $G$ be a group and let $H$ be a subgroup of $G$. Consider the set $A := \{gH : g \in G\}$ of all left cosets. Show that the map $\star$ from $A \times A \to A$ given by $(aH, bH) \mapsto abH$ is well-defined if and only if $H \triangleleft G$.

3. Prove or Disprove that any group group of order $5^2 \cdot 13^2$ is abelian.

4. Decide if the given statement is true or false. If true, prove it. If false, give a counterexample.
   Let $H$ and $K$ be subgroups of the group $G$. If $H \triangleleft K$ and $K \triangleleft G$, then $H \triangleleft G$.

5. We consider the symmetric group $S_5$.
   (a) Let $\alpha = (1 \, 2 \, 3 \, 4 \, 5) \in S_5$. Prove that the centralizer of $\alpha$ in $S_5$ is equal to the cyclic group generated by $\alpha$.
   (b) Prove that the 5-cycles in $A_5$ form two conjugacy classes.

6. Prove that an infinite simple group cannot have a subgroup of finite index.

7. Let $p, q, r$ be distinct odd primes. Describe all ring homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}/pqr\mathbb{Z}$.
   [Hint: You may describe them in terms of the image, $\phi(1)$.]

8. Show that the factor ring $\mathbb{Z}[\sqrt{-2}]/(3 + 2\sqrt{-2})$ is a field.
   [Hint: You may use without proof that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean Domain]
9. Decide which of the following ideals are prime and which are maximal.

   (a) \((x, 4)\) in \(\mathbb{Z}[x]\).
   (b) \((x, y)\) in \(\mathbb{Z}[x, y]\).
   (c) \((3x)\) in \(\mathbb{Z}[x]\).
   (d) \((x, y)\) in \(\mathbb{F}_5[x, y]\).

10. Let \(R\) be a commutative ring with 1. Let

\[ \mathcal{N} = \{ r \in R \mid r^n = 0 \text{ for some } n \in \mathbb{N} \} \]

Show that \(\mathcal{N}\) is contained in the intersection of all prime ideals of \(R\).