Math 519 Qualifying Exam

Spring 2023

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let G be a group. Denote by $Aut(G)$ the group of all automorphisms of G and by $Inn(G)$ its subgroup consisting of all inner automorphisms (i.e., automorphisms of the form $x \mapsto yxy^{-1}$ for some $y \in G$. Consider a map $j : G \to G$ defined by $j(g) = g^{-1}$. Prove that $|Inn(G)| = 1$ if and only if $j \in Aut(G)$.

2. Let G be a group and let H be a subgroup of G. Consider the set $A := \{gH : g \in G\}$ of all left cosets. Show that the map $\star$ from $A \times A \to A$ given by $(aH, bH) \mapsto abH$ is well-defined if and only if $H \triangleleft G$.

3. Prove or Disprove that any group group of order $5^2 \cdot 13^2$ is abelian.

4. Decide if the given statement is true or false. If true, prove it. If false, give a counterexample.

   Let H and K be subgroups of the group G. If $H \triangleleft K$ and $K \triangleleft G$, then $H \triangleleft G$.

5. We consider the symmetric group S_5.

   (a) Let $\alpha = (1\,2\,3\,4\,5) \in S_5$. Prove that the centralizer of α in S_5 is equal to the cyclic group generated by α.

   (b) Prove that the 5-cycles in A_5 form two conjugacy classes.

6. Prove that an infinite simple group cannot have a subgroup of finite index.

7. Let p, q, r be distinct odd primes. Describe all ring homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}/pqr\mathbb{Z}$.

   [Hint: You may describe them in terms of the image, $\phi(1)$.]

8. Show that the factor ring $\mathbb{Z}[\sqrt{-2}]/(3 + 2\sqrt{-2})$ is a field.

   [Hint: You may use without proof that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean Domain]

9. Decide which of the following ideals are prime and which are maximal.

   (a) $(x, 4)$ in $\mathbb{Z}[x]$.
   (b) (x, y) in $\mathbb{Z}[x, y]$.
   (c) $(3x)$ in $\mathbb{Z}[x]$.
   (d) (x, y) in $\mathbb{F}_5[x, y]$.

10. Let R be a commutative ring with 1. Let

$$\mathcal{N} = \{r \in R \mid r^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

Show that $\mathcal{N}$ is contained in the intersection of all prime ideals of R.