High School Mathematics Field Day Exam

Tuesday, March 26, 2024

Southern Illinois University Carbondale

PLEASE READ THESE DIRECTIONS CAREFULLY!

1. Calculators, cell phones, iPods or any other electronic devices ***ARE NOT*** permitted. **Any calculator, cell phone, iPod or any other electronic device seen at any time after entering the auditorium will immediately disqualify that student.**
2. No one is allowed to leave the arena until 11:15 a.m.
3. Use a pencil with No. 2 lead for marking the answer sheet. Only marks in the small circles on the answer sheet are recorded. Avoid smudges. Keep all pencil marks inside the little circles. Do not try to cover up the letter. If the letter is not fully inside the circle it is crucial that you identify and mark the correct circle.
4. a) Follow oral instructions to enter your name.
   b) Enter your four-digit registration number, the number on your 3 x 5 card, under columns JKLM. Then blacken the corresponding numbers underneath.
   c) Failure to correctly fill in your code number is grounds for disqualification.
5. Indicate your answers on the answer sheet. Notice that the scantrons are structured differently than in the past. Check that the number of your answer correctly matches the number of the question. No credit will be given for answers written in this examination booklet. Be sure that each mark on the answer sheet is blackened and **completely fills the circle.** Give only one answer to each question. No credit will be given for multiple answers. If you change an answer, be sure that all previous marks are completely erased. Avoid accidental marks in any other space.
6. Your score will be the number of correct answers marked minus one fourth of the number incorrect answers marked. This is to discourage guessing.
7. Do not spend too much time on any one question. Answer the easier questions first and then go back to the others if time permits.
8. Many questions, even those which look unfamiliar to you, require no specialized mathematical knowledge beyond 9th grade algebra. The questions are not in order of difficulty. If you get stuck, go on; there may be easier questions for you further on.
9. There are 2 blank pages at the end for scratch work.
10. **Begin the test only when told to do so.**
11. You may take the questions home with you.
12. Check the following carefully:
    a) Name is encoded according to the <u>oral instructions</u>.
    b) Your 4-digit registration number is coded. (This is the number on your 3 x 5 card.)
    c) Only one answer per question.
    d) There are 67 problems typed on 14 pages. The test is double-sided.

1. Three successive rolls of a single die all produce a number greater than 3. What is the probability that all show the same number?

a) 1/6 b) 1/9 c) 1/12 d) 1/15 e) none of these

2. If tan 10° = x what is the value of tan 5° in terms of x?

a) $\frac{x}{2}$ b) $\frac{2x}{1-x^2}$ c) $\frac{-1 \pm\sqrt{1+x^2}}{x}$ d) $\frac{-1+\sqrt{1+x^2}}{x}$ e) none of these

3. Simplify $\sqrt{18^3 - 6(18)^2 + 12(18) - 8}$

a) 56 b) 54 c) 64 d) 48 e) none of these

4. Let $i = \sqrt{-1}$. Solve for the complex values of z such that

$$(z + i)^4 = (z - i)^4$$

a) 0 b) $1, -1$ c) $0, 1, -1$ d) $i + 1, i - 1$ e) none of these

5. What is the smallest positive integer which can be expressed as the sum of two distinct prime numbers in exactly three ways?

a) 22 b) 24 c) 26 d) 28 e) none of these

6. Simplify $\sin^{-1}(\sin\left(\frac{3\pi}{4}\right))$

a) 0 b) $\frac{\pi}{4}$ c) $\frac{\sqrt{2}}{2}$ d) $\frac{3\pi}{4}$ e) none of these|

7. A four-digit pin number consists of four digits from the set {0,1,2,3,4,5,6,7,8,9}. How many four-digit pin numbers contain the digit 9 exactly twice?

   a) 5040 b) 3024 c) 504 d) 432 e) none of these

8. An isosceles trapezoid has bases of lengths 8 and 20 and a perimeter of 48. What is the area of this trapezoid?

   a) 112 b) 108 c) 98 d) 122 e) none of these

9. If $\tan^{-1} x = \tan^{-1} 4 + \tan^{-1} 6$ then x equals

   a) $\frac{10}{23}$ b) $\frac{5}{12}$ c) $\frac{-5}{12}$ d) $\frac{-10}{23}$ e) none of these

10. If $a\sqrt{2} + b\sqrt{3} = 0$ and $ab \neq 0$ what is the value of $\frac{a}{b} + \frac{b}{a}$?

    a) $\sqrt{2} + \sqrt{3}$ b) $\sqrt{3} - \sqrt{2}$ c) $\frac{12-\sqrt{2}}{5}$ d) $\frac{-5\sqrt{6}}{6}$ e) none of these

11. A triangle has its sides in a ratio of 6:8:9. This triangle

    a) is an acute triangle b) is a right triangle c) is an obtuse triangle

    d) cannot be uniquely determined e) none of these

12. The real sequence of positive values $\{a, b, c, d\}$ is both arithmetic and geometric. What is the value of $(a + d)^2 - (b + c)^2$

    a) greater than zero b) zero c) less than zero d) imaginary

    e) not uniquely determined

13. How many of the positive integers less than 1001 are a multiple of two or nine?

a) 555 b) 155 c) 611 d) 556 e) none of these

14. Given $\frac{x}{y} = \frac{1+\sqrt{5}}{2}$, what is the value of $\frac{x+y}{y}$?

a) $\frac{1+\sqrt{5}}{2}$ b) $\frac{2+\sqrt{5}}{2}$ c) $\frac{3+\sqrt{5}}{2}$ d) 1 e) none of these

15. If $\cos\theta = \frac{1}{5}$ what is the value of $\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n}$?

a) $\frac{3}{7}$ b) $\frac{4}{7}$ c) $\frac{5}{7}$ d) $\frac{6}{7}$ e) none of these

16. There are exactly four complex numbers z, which satisfy the equation $z^4 = 1$. What is the product of these four numbers?

a) -1 b) 0 c) 1 d) i e) none of these

17. The amounts of electricity bills for all households in a particular city have an approximately normal distribution with a mean of \$140 and a standard of deviation of \$30. Let $\bar{x}$ be the mean for a random sample of 25 homes in this city. What is the standard of deviation for $\bar{x}$?

a) 0 b) 6 c) 30 d) 140 e) none of these

18. Some 30% acid solution and some 50% acid solutions are mixed to produce 400 milliliters of 42% acid solution. How much 50% acid solution must be used?

a) 160 b) 240 c) 220 d) 180 e) none of these

19. Three points A,B and C are placed at random on a circle. The probability that all three points lie on a semi-circle is

a) $\frac{3}{4}$ b) $\frac{1}{2}$ c) $\frac{1}{3}$ d) $\frac{2}{5}$ e) none of these

20. The expression $(\sqrt{3} + \sqrt{7})^2$ can be written as x + y$\sqrt{14}$, where x is the integer 10. What is the value of x + y^2?

a) 18 b) 22 c) 16 d) 12 e) none of these

21. Half of a half is

a) 1 b) $\frac{1}{2}$ c) $\frac{1}{4}$ d) $\frac{1}{3}$ e) none of these

22. Two coplanar circles have radii of 4cm and 12cm. If the centers are 17 cm apart, what is the length of the common external tangent in cm?

a) $\sqrt{145}$ b) $\sqrt{433}$ c) 17 d) 15 e) none of these

23. If five cats catch five rats in ten minutes, how long will it take ten cats to catch ten rats? (assume all cats catch rats at the same rate)

a) 5 minutes b) 10 minutes c) 20 minutes d) 40 minutes e) none of these

24. If the number 196_x (one nine six in base x) equals 342_8, what is the value of x?

a) 9 b) 10 c) 11 d) 12 e) none of these

25. If a six-sided red die and a six-sided green die, each numbered one through six are rolled, what is the probability the sum of the top numbers showing is a seven given the red die shows an odd number?

a) $\frac{1}{5}$ b) $\frac{1}{6}$ c) $\frac{1}{8}$ d) $\frac{1}{9}$ e) none of these

26. How many real roots does the equation $x^7 + 3x^5 + 8x^3 + 12x = 0$ have?

a) 0 b) 1 c) 5 d) 7 e) none of these

27. If $a < b$, the roots of $x^2 + ax + b = 0$ are $a + 10$ and $b - 8$. What are the values of a and b?

a) $a = -6, b = 10$ b) $a = -2, b = 3$ c) $a = 4, b = 6$

d) $a = -7, b = 12$ e) none of these

28. The points (0,0), (2,5), and (6,0) are three vertices of a parallelogram with (a,b) being the fourth vertex. The smallest possible value of a + b is

a) 13 b) -1 c) 1 d) -7 e) none of these

29. Ten out of the 250 cars selected at random manufactured at a plant were found to be defective. Assume that the defective cars are manufactured randomly. What is the probability that the next car manufactured is not defective?

a) 0 b) 0.04 c) 0.96 d) 1 e) none of these

30. Let f (x) be a function such that for x $\neq 2$ that $f\left(\frac{x+3}{x-2}\right) = \frac{3x-6}{2x+6}$. What is f(x)?

a) $f(x) = \frac{1}{x}$ b) $f(x) = \frac{x+1}{x-2}$ c) $f(x) = \frac{x^2+12}{x^2-1}$ d) $f(x) = \frac{3}{2x}$ e) $f(x) = \frac{2}{x}$

31. The greatest integer function $[\![x]\!]$ is defined to be the largest integer less than or equal to x. Solve $[\![x]\!]^2 + 2\,[\![x]\!] - 15 = 0$ Answers are given in interval notation

a) (2,3] or $(-6,-5]$ b) [3,4) or $[-5,-4)$

c) [2,3) or $[-6,-5)$ d) (3,4] or $(-5,-4]$ e) none of these

32. The sum of the distinct prime factors dividing 2024 is

a) 36 b) 39 c) 42 d) 255 e) none of these

33. Find the distance between the points of intersection of the circles with equations
$(x-1)^2 + (y+2)^2 = 4$ and $(x+1)^2 + (y+1)^2 = 1$

a) $\frac{3\sqrt{2}}{2}$ b) 5 c) $\frac{4\sqrt{5}}{5}$ d) $2\sqrt{5}$ e) none of these

34. The positive difference between the repeating decimal $0.kkk\ldots_n$ and the terminating decimal $0.k_n$ where both decimals are represented in base n where $n > k > 1$ is

a) $\frac{k}{n}$ b) $\frac{k}{n^2-n}$ c) $\frac{k+1}{n^2-n}$ d) $\frac{2k}{n^2+n}$ e) none of these

35. Anna was born November 22, and interestingly 11 and 22 are her favorite numbers. At her work employees are assigned a 4-digit extension at random. These extensions can contain any digit from 0 to 9 inclusive and repeated digits are allowed. What is the probability Anna's extension consists of the digits 1,1,2,2 in any order?

a) $\frac{1}{10^3}$ b) $\frac{1}{10^4}$ c) $\frac{4!}{10 \cdot 9 \cdot 8 \cdot 7}$ d) $\frac{6}{10^4}$ e) none of these

36. If A,B,C,D are distinct points with the distance between points A and B equal to x, where $x > 0$. C is the midpoint of segment AB. The distance from C to point D is $\frac{x}{2}$. What is the measure of angle ADB?

a) 45° b) 60° c) 90° d) 120° e) none of these

37. Find the value of $a + b$ if (a, b) is the solution to the system of equations

$$5^{a+1} = 125^b \quad \text{and} \quad 9^b = 3^{a-2}$$

a) -5 b) 0 c) 3 d) 13 e) none of these

38. A quadratic equation with real integral roots has the sum of its roots equal to s and the product of its roots equal to p. How is s related to p?

a) two integers that add to s must multiply to get p

b) the opposite of the two integers which add to s must multiply to get p

c) two integers which add to p must multiply to get s

d) the opposite of the two integers which add to p must multiply to get s

e) none of these

39. Solve for x if: log(log(log(log(x)))) = 0.

a) 0 b) 10^{10} c) 10^{100} d) $10^{10^{10}}$ e) none of these

40. Simplify the following expression if x≠ 0. $(\frac{7x^2}{x^3})^2(4x^2)^0$

a) $\frac{49}{x}$ b) $\frac{196}{x}$ c) 196x d) 49x e) none of these

41. To what form can the expression $(2x-3)^2-(2x+3)^2$ be simplified to?

a) 0 b) 18 c) $8x^2-24x$ d) $-24x$ e) none of these

42. The sum of the solutions to $|2x-3|=|x+9|$

a) 0 b) 12 c) 8 d) 10 e) none of these

43. Simplify $\sqrt{5-\sqrt{21}}$

a) $\sqrt{14}-\sqrt{6}$ b) $\frac{\sqrt{14}-\sqrt{6}}{2}$ c) $\sqrt{14}+\sqrt{6}$ d) $\frac{\sqrt{14}+\sqrt{6}}{2}$ e) none of these

44. Evaluate $\prod_{n=1}^{13}\frac{n(n+2)}{(n+4)^2}$

a) $\frac{3}{16140}$ b) $\frac{3}{161840}$ c) $\frac{3}{16840}$ d) $\frac{3}{151840}$ e) none of these

45. For what value of k do both the expressions $x^2 + kx + 48$ and $x^2 + 20x + k$ factor?

a) 21 b) 24 c) 19 d) 26 e) none of these

46. Solve for x if $(\sqrt{5}+2)^x + (\sqrt{5}-2)^x = 18$

a) 2 b) -2 c) $2, -2$ d) 3 e) none of these

47. The expression $(3x + 4y - 3z)^5$ is multiplied out and the sum of the coefficients of every term is computed. This sum is

a) less than 500 b) between 500 and 1000 c) between 1000 and 2000

d) between 2000 and 4000 e) more than 4000

48. Let a, b, c be the roots of $x^3 - 2x^2 - 6x + 10 = 0$. Find the value of the expression

$$\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}.$$

a) $\frac{-1}{5}$ b) $\frac{2}{5}$ c) $\frac{7}{8}$ d) $\frac{-3}{10}$ e) none of these

49. Evaluate the expression $\frac{53^3 + 24^3}{53^3 + 29^3}$

a) $\frac{77}{82}$ b) $\frac{75}{84}$ c) $\frac{81}{86}$ d) $\frac{91}{96}$ e) none of these

50. Solve x if $\sqrt{x} + \sqrt{x-11} - 11 = 0$

a) 6 b) 36 c) 49 d) 16 e) none of these

51. Find the sum of the integers not in the domain of the function
f(x) = $\sqrt{x^2 - 8x + 12}$

a) 20 b) 12 c) 60 d) 16 e) none of these

52. A cube of wood measures six inches on each side. This cube is dipped in orange paint so that the outside of the cube is now painted orange. After the paint dries it is then cut uniformly into one-inch cubes. These cubes are then placed into a box and one cube is drawn at random. What is the probability this cube has exactly two sides which are orange?

a) $\frac{1}{9}$ b) $\frac{2}{3}$ c) $\frac{1}{3}$ d) $\frac{1}{6}$ e) none of these

53. An equilateral triangle has its perimeter numerically equal to its area. What is its perimeter?

a) $2\sqrt{3}$ b) $4\sqrt{3}$ c) $6\sqrt{3}$ d) $12\sqrt{3}$ e) none of these

54. Find the smallest value of x + y if $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$ and $x - y = 3$.

a) -1 b) -4 c) 10 d) 9 e) none of these

55. If (x, y) is the intersection of the curves $y = x^4 + 4x^3 + 7x^2 + 2x + 2$ and $y = (x + 1)^4$ what is the value of $4x + y$?

a) 17 b) 20 c) 12 d) 22 e) none of these

56. If xy = 11, yz = 22 and xz = 2, what is the value of $\frac{x+y}{z}$?

a) 5 b) 6 c) ± 6 d) ± 4 e) none of these

57. If the average of a and b is 5, the average of c,d and f is 10, and g = 20, what is the average of a,b,c,d,f and g ?

a) $\frac{35}{6}$ b) 35 c) 10 d) 12 e) none of these

58. Four friends are playing cards in the lounge of a hotel. Each has their own room and the keys to the rooms are all identical. They put their keys in the mailbox to one of their rooms all in order. Not known to the friends playing cards, a child sees the room keys and takes them out and plays with them, totally mixing up their order. The child then puts the keys back where they found them. What is the probability that exactly two of the friends has the correct key when they return to their room?

a) $\frac{2}{3}$ b) $\frac{1}{3}$ c) $\frac{1}{6}$ d) $\frac{1}{4}$ e) none of these

59. What is the value of x^2 if $2(x-3) = 3x - 15$?

a) 81 b) 9 c) 144 d) 12 e) none of these

60. Solve for x if $5^x 7^{x^2} = 35$.

a) x =1 b) x = log 35 c) x =1 or x = $\frac{-\log 35}{\log 7}$ d) x = $\frac{\log(\frac{5}{7})}{\log 7}$

e) none of these

61. Simplify $\sqrt{2.777\ldots}$

a) $\frac{5}{3}$ b) $\frac{\sqrt{7}}{3}$ c) $\frac{27}{10}$ d) $\frac{2}{3}$ e) none of these

62. Simplify $\frac{1}{sin10°} - \frac{\sqrt{3}}{cos10°}$

a) 2 b) 1 c) -1 d) 4 e) none of these

63. Let triangle ABC be a right triangle with angle B as the right angle. Side AC is 28 and the area of the triangle is 98. If it is the smallest angle in the triangle, what is the measure of angle BCA in radians?

a) $\frac{\pi}{6}$ b) $\frac{\pi}{12}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{3}$ e) none of these

64. When written in descending powers of r, what is the twelfth term of $(2r - 5s)^{18}$? (note: $\binom{18}{7}$ represents the number of combinations of seven objects selected from 18 objects)

a) $\binom{18}{7}(2r)^7(-5s)^{11}$ b) $\binom{18}{11}(2r)^6(-5s)^{12}$ c) $\binom{18}{7}(2r)^7(5s)^{11}$

d) $\binom{18}{7}(2r)^7(-5s)^{12}$ e) none of these

65. In a kindergarten class there are 20 students of which there are two sets of identical twins. If they line up for recess at random, what is the probability that both sets of twins are next to each other.

a) $\frac{1}{95}$ b) $\frac{1}{90}$ c) $\frac{2}{95}$ d) $\frac{4}{85}$ e) none of these

66. For what values of a, does the line $y = -3$ intersect the parabola $y = x^2 + 10x + a$ in exactly one point?

a) 11 or 2 b) 22 c) 88 d) 7 or 9 e) none of these

67. Simplify $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$

a) 1 b) $\sqrt[3]{5}$ c) $3-\sqrt{5}$ d) $4-\sqrt{5}$ e) none of these